Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Approximation Theory 120 (2003) 185–216
JOURNAL OF Approximation Theory
http://www.elsevier.com/locate/jat

Approximation by polynomials and smooth functions in Sobolev spaces with respect to measures

José M. Rodríguez[1]

Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III de Madrid, Avenida de la Universidad, 30 28911 Leganés, Madrid, Spain

Received 18 January 2002; accepted in revised form 30 September 2002

Abstract

The density of polynomials is straightforward to prove in Sobolev spaces $W^{k,p}((a,b))$, but there exist only partial results in weighted Sobolev spaces; here we improve some of these theorems. The situation is more complicated in infinite intervals, even for weighted L^p spaces; besides, in the present paper we have proved some other results for weighted Sobolev spaces in infinite intervals.

Keywords: Polynomial approximation; Weighted Sobolev spaces; Sobolev spaces with measures

1. Introduction

Weighted Sobolev spaces are an interesting topic in many fields of Mathematics (see e.g. [HKM,K,Ku,KO,KS,T]). In [EL,ELW1,ELW2] the authors study some examples of Sobolev spaces with respect to general measures instead of weights, in relation with ordinary differential equations and Sobolev orthogonal polynomials. The papers [R1,R2,RARP1,RARP2] are the beginning of a theory of Sobolev spaces with respect to general measures. We are interested in the relationship between this topic and Approximation Theory in general, and Sobolev orthogonal polynomials in particular.

E-mail address: jomaro@math.uc3m.es.
[1] Research partially supported by a Grant from DGI (BFM 2000-0206-C04-01), Spain.

0021-9045/02/$ - see front matter
PII: S0021-9045(02)00019-9

Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, \ldots, \mu_k)$ a vectorial Borel measure in $\mathbf{R}$ with $\Delta := \cup_{j=0}^{k} \operatorname{supp} \mu_j$. The Sobolev norm of a function f of class $C^k(\mathbf{R})$ in $W^{k,p}(\Delta, \mu)$ is defined by

$$||f||^p_{W^{k,p}(\Delta,\mu)} := \sum_{j=0}^{k} \int |f^{(j)}|^p \, d\mu_j.$$

We talk about Sobolev norm although it can be a seminorm; in this case we will take equivalence classes, as usual.

If every polynomial belongs to $L^p(\mu_0) \cap L^p(\mu_1) \cap \cdots \cap L^p(\mu_k)$, we denote by $P^{k,p}(\Delta, \mu)$ the completion of polynomials P with the norm of $W^{k,p}(\Delta, \mu)$. By a theorem in [LP] we know that, if Δ is a compact set, the zeroes of the Sobolev orthogonal polynomials with respect to the scalar product in $W^{k,2}(\Delta, \mu)$ are contained in the disk $\{z : |z| \leqslant 2||M||\}$, where the multiplication operator $(Mf)(x) = xf(x)$ is considered in the space $P^{k,2}(\Delta, \mu)$. Consequently, the set of the zeroes of the Sobolev orthogonal polynomials is bounded if the multiplication operator is bounded. The location of these zeroes allows to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [LP]).

In [R2,RARP2] there are necessary conditions and sufficient conditions for M to be bounded. A fundamental tool in this work is to know what are the elements in $P^{k,2}(\Delta, \mu)$. In fact, this is a central problem in Approximation Theory: find the class of functions which can be approximated by polynomials or smooth functions with a given norm. If Δ is a compact set, it is equivalent to approximate by polynomials or $C^\infty(\mathbf{R})$ functions, since the Bernstein's proof of Weierstrass' Theorem (see e.g. [D, p. 113]) gives that every function in $C^k([a, b])$ can be approximated by polynomials uniformly up to the kth derivative. However, if Δ is non-compact it is more difficult to approximate functions by polynomials than by functions in $C^\infty(\mathbf{R})$.

In [RARP2, Theorem 4.1], there are sufficient conditions in order to have $P^{k,p}(\Delta, \mu) = W^{k,p}(\Delta, \mu)$, if we define in a correct way these Sobolev spaces. In this paper, we obtain improvements of Theorem 4.1 in [RARP2] in the case of Δ compact and new results for the non-compact case. Observe that Theorem 4.3 in [RARP2] (see Theorem E) gives a criterion to obtain the density of smooth functions in the non-compact case, but it cannot be applied to have the density of polynomials.

Now, let us state the main results here. We refer to the definitions in the next section. In the paper, the results are numbered according to the section where they are proved.

First, we have four theorems which give sufficient conditions for $C_c^\infty(\mathbf{R})$ to be dense in $W^{k,p}(\Delta, \mu)$, if Δ is a compact interval. Observe that under this hypothesis on Δ, $C_c^\infty(\mathbf{R})$ is dense if and only if $C^\infty(\mathbf{R})$ or P is dense.

Theorem 3.1. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, \mu_1)$ a finite p-admissible vectorial measure with $\Delta = [a, b]$ and $w_1 := d\mu_1/dx \in B_p((a, b))$. Then $C_c^\infty(\mathbf{R})$ is dense in $W^{1,p}([a, b], \mu)$.*

Theorem 3.2. *Let us consider* $1\leqslant p<\infty$, $0<m\leqslant k$ *and* $\mu=(\mu_0,\ldots,\mu_k)$ *a finite p-admissible vectorial measure with* $\Delta=[a,b]$ *and* $C_c^\infty(\mathbf{R})$ *dense in* $W^{k-m,p}([a,b],\mu)$. *Assume that* $([a,b],(\mu_m,\ldots,\mu_k))\in\mathscr{C}_0$ *if* $m<k$. *Assume also that we have either*:

(1) $\Omega^{(m-1)}=[a,b]$,
(2) $\Omega^{(m-1)}=(a,b]$ *and there exists* $\varepsilon>0$ *such that* $\mu|_{[a,a+\varepsilon]}$ *is a right completion of the vectorial measure* $(0,\ldots,0,\mu_m,\ldots,\mu_k)$.

Then $C_c^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([a,b],\mu)$.

Theorem 3.3. *Let us consider* $1\leqslant p<\infty$ *and* $\mu=(\mu_0,\ldots,\mu_k)$ *a finite p-admissible vectorial measure with* $\Delta=[a,b]$ *and* $w_k:=d\mu_k/dx\in B_p((a,b))$. *Assume that we have either*:

(1) *a is right* $(k-2)$*-regular if* $k\geqslant 2$,
(2) *there exists* $\varepsilon>0$ *such that* $\mu|_{[a,a+\varepsilon]}$ *is a right completion of* $(0,\ldots,0,\mu_{k-1},\mu_k)$ if $k\geqslant 2$.

Then $C_c^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([a,b],\mu)$.

Theorem 3.4. *Let us consider* $1\leqslant p<\infty$, *a compact interval I and a finite p-admissible vectorial measure* $\mu=(\mu_0,\ldots,\mu_k)$ *with* $\Delta=I$. *Assume that there exist* $a_0\in I$, *an integer* $0\leqslant r<k$ *and positive constants* c,δ *such that*

(1) $d\mu_{j+1}(x)\leqslant c\,|x-a_0|^p\,d\mu_j(x)$ *in* $[a_0-\delta,a_0+\delta]\cap I\quad$ *for* $r\leqslant j<k$,
(2) $w_k:=d\mu_k/dx\in B_p(I\setminus\{a_0\})$,
(3) *if* $r>0$, a_0 *is* $(r-1)$*-regular*.

Then $C_c^\infty(\mathbf{R})$ *is dense in* $W^{k,p}(I,\mu)$.

The following result gives a sufficient condition for P to be dense in $W^{k,p}(\Delta,\mu)$, without hypothesis on the support Δ.

Theorem 4.1. *Let us consider* $1\leqslant p<\infty$ *and* $\mu=(\mu_0,\ldots,\mu_k)$ *a p-admissible vectorial measure. Assume that there exist* $a\in\Delta$ *and a positive constant c such that*

$$c||g||_{W^{k,p}(\Delta,\mu)}\leqslant|g(a)|+|g'(a)|+\cdots+|g^{(k-1)}(a)|+||g^{(k)}||_{L^p(\Delta,\mu_k)}$$

for every $g\in V^{k,p}(\Delta,\mu)$. *Then, P is dense in* $W^{k,p}(\Delta,\mu)$ *if and only if P is dense in* $L^p(\Delta,\mu_k)$.

As an application of this theorem we can obtain many results for particular weights. We make the computations for the following cases: Laguerre, Freud and weights of fast decreasing degree.

Proposition 4.1. *Consider* $1\leqslant p<\infty$ *and a vectorial weight* w *in* $(0,\infty)$, *with*

(1) $w_j(x)\leqslant c_j x^{\beta_j}$, *for* $0\leqslant j<k$, $w_k(x)\geqslant c_k x^{\beta_k}$, *in* $(0,a)$,
(2) $w_j(x)\leqslant c_j x^{\alpha+(k-j)(\varepsilon-1)p}e^{-\lambda x^{\varepsilon}}$, *for* $0\leqslant j<k$, $w_k(x)\geqslant c_k x^{\alpha}e^{-\lambda x^{\varepsilon}}$, *in* (a,∞),
where $\alpha\in\mathbf{R}$, $a,\varepsilon,\lambda,c_j>0$ *and* $\beta_j>-1$, *for* $0\leqslant j\leqslant k$. *Then the polynomials are dense in* $W^{k,p}([0,\infty),w)$ *if they are dense in* $L^p((0,\infty),w_k)$ *and* $\beta_j\geqslant\beta_k-(k-j)p$, *for* $0\leqslant j<k$.

Corollary 4.1. *Consider* $1\leqslant p<\infty$ *and a vectorial weight* w, *with* $w_j(x)\asymp x^{\alpha_j}e^{-\lambda x^{\varepsilon}}$ *in* $(0,\infty)$, *for* $0\leqslant j\leqslant k$, *where* $\varepsilon\geqslant\frac{1}{2}$, $\lambda>0$ *and* $\alpha_j>-1$, *for* $0\leqslant j\leqslant k$. *Then the polynomials are dense in* $W^{k,p}([0,\infty),w)$ *if* $\alpha_k-(k-j)p\leqslant\alpha_j\leqslant\alpha_k+(k-j)(\varepsilon-1)p$, *for* $0\leqslant j<k$.

Proposition 4.2. *Consider* $1\leqslant p<\infty$ *and a vectorial weight* w *in* $\mathbf{R}$, *with*

(1) $w_j(x)\leqslant c_j|x|^{\alpha+(k-j)(\varepsilon-1)p}e^{-\lambda|x|^{\varepsilon}}$, *for* $0\leqslant j<k$, $w_k(x)\geqslant c_k|x|^{\alpha}e^{-\lambda|x|^{\varepsilon}}$, *in* (B,∞),
(2) $w_j(x)\leqslant c_j|x|^{\alpha'+(k-j)(\varepsilon'-1)p}e^{-\lambda'|x|^{\varepsilon'}}$, *for* $0\leqslant j<k$, $w_k(x)\geqslant c_k|x|^{\alpha'}e^{-\lambda'|x|^{\varepsilon'}}$, *in* $(-\infty,-A)$,
(3) $w_j(x)\in L^1([-A,B])$, *for* $0\leqslant j\leqslant k$, $w_k(x)\in B_p([-A,B])$,
where $\alpha,\alpha'\in\mathbf{R}$, $\varepsilon,\varepsilon'\geqslant 1$ *and* $A,B,\lambda,\lambda',c_j>0$, *for* $0\leqslant j\leqslant k$. *Then the polynomials are dense in* $W^{k,p}(\mathbf{R},w)$ *if they are dense in* $L^p(\mathbf{R},w_k)$.

Corollary 4.2. *Consider* $1\leqslant p<\infty$ *and a vectorial weight* w *in* $\mathbf{R}$, *with* $w_j(x)\asymp|x|^{\alpha_j}e^{-\lambda|x|^{\varepsilon}}$ *in* $\mathbf{R}$, *for* $0\leqslant j\leqslant k$, *where* $\varepsilon\geqslant 1$, $\lambda>0$ *and* $\alpha_j>-1$, *for* $0\leqslant j\leqslant k$. *Assume also that* $\alpha_k<p-1$ *if* $p>1$, *and* $\alpha_k\leqslant 0$ *if* $p=1$. *Then the polynomials are dense in* $W^{k,p}(\mathbf{R},w)$ *if* $\alpha_j\leqslant\alpha_k+(k-j)(\varepsilon-1)p$, *for* $0\leqslant j<k$.

Proposition 4.3. *Consider* $1\leqslant p<\infty$ *and a vectorial weight* w, *with* $w_j(x)\leqslant c_j\exp_{-\lambda_1,\lambda_2,\ldots,\lambda_n}(|x|^{\varepsilon})$ *in* $\mathbf{R}$, *for* $0\leqslant j<k$, $w_k(x)\geqslant c_k\exp_{-\lambda_1,\lambda_2,\ldots,\lambda_n}(|x|^{\varepsilon})$ *in* $\mathbf{R}$, *where* $n>1$ *and* $\varepsilon,\lambda_1,\lambda_2,\ldots,\lambda_n,c_0,c_1,\ldots,c_k>0$. *Then the polynomials are dense in* $W^{k,p}(\mathbf{R},w)$ *if they are dense in* $L^p(\mathbf{R},w_k)$.

Corollary 4.3. *Consider* $1\leqslant p<\infty$ *and a vectorial weight* w, *with* $w_j(x)\asymp\exp_{-\lambda_1,\lambda_2,\ldots,\lambda_n}(|x|^{\varepsilon})$ *in* $\mathbf{R}$, *for* $0\leqslant j\leqslant k$, *where* $n>1$ *and* $\varepsilon,\lambda_1,\lambda_2,\ldots,\lambda_n>0$. *Then the polynomials are dense in* $W^{k,p}(\mathbf{R},w)$.

We also obtain results which allow to decide in many cases when two norms are comparable. Now we present the notation we use.

Notation: In the paper $k\geqslant 1$ denotes a fixed natural number; obviously $W^{0,p}(\Delta,\mu)=L^p(\Delta,\mu)$. All the measures we consider are Borel and positive on $\mathbf{R}$; if a measure is defined in a proper subset $E\subset\mathbf{R}$, we define it in $\mathbf{R}\setminus E$ as the zero measure. Also, all the weights are non-negative Borel measurable functions defined on $\mathbf{R}$. If the measure does not appear explicitly, we mean that we are using Lebesgue measure. We always work with measures which satisfy the decomposition $d\mu_j=d(\mu_j)_{\mathrm{s}}+d(\mu_j)_{\mathrm{ac}}=d(\mu_j)_{\mathrm{s}}+h\,dx$, where $(\mu_j)_{\mathrm{s}}$ is singular with respect to Lebesgue

measure, $(\mu_j)_{\mathrm{ac}}$ is absolutely continuous with respect to Lebesgue measure and h is a Lebesgue measurable function (which can be infinite in a set of positive Lebesgue measure); obviously every σ-finite measure belongs to this class. Given $0<m<k$, a vectorial measure μ and a closed set E, we denote by $W^{k,p}(E,\mu)$ the space $W^{k,p}(\Delta\cap E,\mu|_E)$ and by $W^{k-m,p}(\Delta,\mu)$ the space $W^{k-m,p}(\Delta,(\mu_m,\ldots,\mu_k))$. We denote by $\operatorname{supp} v$ the support of the measure v. If A is a Borel set, $|A|$, χ_A, $\bar{A}$ and $\#A$ denote, respectively, the Lebesgue measure, the characteristic function, the closure and the cardinality of A. By $f^{(j)}$ we mean the jth distributional derivative of f. P denotes the set of polynomials. We say that an n-dimensional vector satisfies an one-dimensional property if each coordinate satisfies this property. Finally, the constants in the formulae can vary from line to line and even in the same line.

The outline of the paper is as follows. Section 2 presents most of the definitions we need to state our results; we also collect the technical results of [R2,RARP1,RARP2] that we need. Section 3 is dedicated to the proof of the theorems on density for measures with compact support. In Section 4 we prove the theorems on density without hypothesis on the support. We prove some results on comparable norms in Sobolev spaces in Section 5.

2. Definitions and previous results

Obviously, one of our main problems is to define correctly the space $W^{k,p}(\Delta,\mu)$. There are two natural definitions:

(1) $W^{k,p}(\Delta,\mu)$ is the biggest space of (classes of) functions f which are regular enough to have $\|f\|_{W^{k,p}(\Delta,\mu)}<\infty$.
(2) $W^{k,p}(\Delta,\mu)$ is the closure of a good set of functions (e.g. $C^\infty(\mathbf{R})$ or P) with the norm $\|\cdot\|_{W^{k,p}(\Delta,\mu)}$.

However, both approaches have serious difficulties:

We consider first approach (1). It is clear that the derivatives $f^{(j)}$ must be distributional derivatives in order to have a complete Sobolev space. Therefore, we need to restrict the measures μ to a class of p-admissible measures (see Definition 8). Roughly speaking μ is p-admissible if $(\mu_j)_{\mathrm{s}}$, for $0<j\leqslant k$, is concentrated in the set of points where $f^{(j)}$ is continuous, for every function f of the space, because otherwise $f^{(j)}$ is determined, up to zero-Lebesgue measure sets. This will force $(\mu_k)_{\mathrm{s}}$ to be identically zero. However, there will be no restriction on the support of $(\mu_0)_{\mathrm{s}}$.

This reasonable approach excludes norms appearing in the theory of Sobolev orthogonal polynomials. Even if we work with the simpler case of the weighted Sobolev spaces $W^{k,p}(\Delta,w)$ (measures without singular part) we must impose that w_j belongs to the class B_p (see Definition 2) in order to have a complete weighted Sobolev space (see [KO,RARP1]).

Approach (2) is simpler: a classical theorem says that the completion of every normed space exists (e.g. $(C^\infty(\mathbf{R}),\|\cdot\|_{W^{k,p}(\Delta,\mu)})$ or $(P,\|\cdot\|_{W^{k,p}(\Delta,\mu)})$). However, we

have two difficulties. The first one is evident: we do not have an explicit description of the Sobolev functions as in (1) (in [RARP2, Section 4] there are several theorems which give that both definitions of Sobolev space are the same for p-admissible measures). The second problem is worse: the completion of a normed space is by definition a set of equivalence classes of Cauchy sequences. In many cases this completion is not a function space (see [R2, Theorem 3.1] and its Remark). Here, we work with the first approach; see [R2] in order to deal with the second one.

First of all, we explain the definition of generalized Sobolev space in [RARP1]. We start with some preliminary definitions.

Definition 1. We say that two functions u, v are comparable on the set A if there are positive constants c_1, c_2 such that $c_1 v(x) \leqslant u(x) \leqslant c_2 v(x)$ for almost every $x \in A$. Since measures and norms are functions on measurable sets and vectors, respectively, we can talk about comparable measures and comparable norms. We say that two vectorial weights or vectorial measures are comparable if each component is comparable.

In what follows, the symbol $a \asymp b$ means that a and b are comparable for a and b functions, measures or norms.

Obviously, the spaces $L^p(A, \mu)$ and $L^p(A, v)$ are the same and have comparable norms if μ and v are comparable on A. Therefore, in order to study Sobolev spaces we can change a measure μ to any comparable measure v.

Next, we shall define a class of weights which plays an important role in our results.

Definition 2. We say that a weight w belongs to $B_p([a, b])$, with $1 \leqslant p < \infty$, if

$$w^{-1} \in L^{1/(p-1)}([a, b]).$$

Also, if J is any interval we say that $w \in B_p(J)$ if $w \in B_p(I)$ for every compact interval $I \subseteq J$. We say that a weight belongs to $B_p(J)$, where J is a union of disjoint intervals $\cup_{i \in A} J_i$, if it belongs to $B_p(J_i)$, for $i \in A$.

Remark. If $d\mu := w\, dx$ in some interval J, with $w \in B_p(J)$, then the Lebesgue measure in J is absolutely continuous with respect to μ.

Observe that if $v \geqslant w$ in J and $w \in B_p(J)$, then $v \in B_p(J)$.

The class $B_p(\mathbf{R})$ contains the classical $A_p(\mathbf{R})$ weights appearing in Harmonic Analysis (see [Mu1] or [GR]). The classes $B_p(\Omega)$, with $\Omega \subseteq \mathbf{R}^n$, and $A_p(\mathbf{R}^n)$ $(1 < p < \infty)$ have been used in other definitions of weighted Sobolev spaces in [K,KO], respectively.

Definition 3. We denote by $AC([a, b])$ the set of functions absolutely continuous in $[a, b]$, i.e. the functions $f \in C([a, b])$ such that $f(x) - f(a) = \int_a^x f'(t)\, dt$ for all $x \in [a, b]$. If J is any interval, $AC_{\mathrm{loc}}(J)$ denotes the set of functions absolutely continuous in every compact subinterval of J.

Definition 4. Let us consider $1 \leqslant p < \infty$ and a vectorial measure $\mu = (\mu_0, \ldots, \mu_k)$ with absolutely continuous part $w = (w_0, \ldots, w_k)$. For $0 \leqslant j \leqslant k$ we define the open set

$$\Omega_j := \{x \in \mathbf{R} : \exists \text{ an open neighbourhood } V \text{ of } x \text{ with } w_j \in B_p(V)\}.$$

Observe that we always have $w_j \in B_p(\Omega_j)$ for any $1 \leqslant p < \infty$ and $0 \leqslant j \leqslant k$. In fact, Ω_j is the greatest open set U with $w_j \in B_p(U)$. Obviously, Ω_j depends on p and μ, although p and μ does not appear explicitly in the symbol Ω_j. Applying Hölder inequality it is easy to check that if $f^{(j)} \in L^p(\Omega_j, w_j)$ with $1 \leqslant j \leqslant k$, then $f^{(j)} \in L^1_{\mathrm{loc}}(\Omega_j)$ and $f^{(j-1)} \in AC_{\mathrm{loc}}(\Omega_j)$.

Hypothesis. From now on we assume that w_j is identically 0 on the complement of Ω_j.

We need this hypothesis in order to have complete Sobolev spaces (see [KO,RARP1]).

Remark. This hypothesis is satisfied, for example, if we can modify w_j in a set of zero Lebesgue measure in such a way that there exists a sequence $\alpha_n \searrow 0$ with $w_j^{-1}\{(\alpha_n, \infty]\}$ open for every n. If w_j is lower semicontinuous, then this condition is satisfied.

The following definitions also depend on μ and p, although μ and p do not appear explicitly.

Let us consider $1 \leqslant p < \infty$, $\mu = (\mu_0, \ldots, \mu_k)$ a vectorial measure and $y \in \Delta$. To obtain a greater regularity of the functions in a Sobolev space we construct a modification of the measure μ in a neighbourhood of y, using the following Muckenhoupt weighted version of Hardy inequality (see [Mu2,M, p. 44]). This modified measure is equivalent in some sense to the original one (see Theorem A).

Muckenhoupt inequality I. *Let us consider $1 \leqslant p < \infty$ and μ_0, μ_1 measures in $(a, b]$ with $w_1 := d\mu_1/dx$. Then there exists a positive constant c such that*

$$\left|\left|\int_x^b g(t)\,dt\right|\right|_{L^p((a,b],\mu_0)} \leqslant c||g||_{L^p((a,b],\mu_1)}$$

for any measurable function g in $(a, b]$, if and only if

$$\sup_{a<r<b} \mu_0((a, r])||w_1^{-1}||_{L^{1/(p-1)}([r,b])} < \infty.$$

Definition 5. A vectorial measure $\bar{\mu} = (\bar{\mu}_0, \ldots, \bar{\mu}_k)$ is a right completion of a vectorial measure $\mu = (\mu_0, \ldots, \mu_k)$ with respect to y, if $\bar{\mu}_k := \mu_k$ and there is an $\varepsilon > 0$ such that $\bar{\mu}_j := \mu_j$ in the complement of $(y, y + \varepsilon]$ and

$$\bar{\mu}_j := \mu_j + \tilde{\mu}_j \quad \text{in } (y, y + \varepsilon] \text{ for } 0 \leqslant j < k,$$

where $\tilde{\mu}_j$ is any measure satisfying:

(i) $\tilde{\mu}_j((y,y+\varepsilon])<\infty$,
(ii) $\Lambda_p(\tilde{\mu}_j,\bar{\mu}_{j+1})<\infty$, with

$$\Lambda_p(v,\sigma):=\sup_{y<r<y+\varepsilon} v((y,r])\left\|\left(\frac{d\sigma}{dx}\right)^{-1}\right\|_{L^{1/(p-1)}([r,y+\varepsilon])}.$$

The Muckenhoupt inequality guarantees that if $f^{(j)}\in L^p(\mu_j)$ and $f^{(j+1)}\in L^p(\mu_{j+1})$, then $f^{(j)}\in L^p(\bar{\mu}_j)$. If we work with absolutely continuous measures, we also say that a vectorial weight $\bar{w}$ is a completion of μ (or of w).

Example. It can be shown that the following construction is always a completion: we choose $\tilde{w}_j:=0$ if $\bar{w}_{j+1}\notin B_p((y,y+\varepsilon])$; if $\bar{w}_{j+1}\in B_p([y,y+\varepsilon])$ we set $\tilde{w}_j(x):=1$ in $[y,y+\varepsilon]$; and if $\bar{w}_{j+1}\in B_p((y,y+\varepsilon])\setminus B_p([y,y+\varepsilon])$ we take $\tilde{w}_j(x):=1$ for $x\in[y+\varepsilon/2,y+\varepsilon]$, and

$$\tilde{w}_j(x):=\begin{cases}\dfrac{d}{dx}\left\{\left(\int_x^{y+\varepsilon}\bar{w}_{j+1}^{-1/(p-1)}\right)^{-p+1}\right\}=\dfrac{(p-1)\,\bar{w}_{j+1}(x)^{-1/(p-1)}}{(\int_x^{y+\varepsilon}\bar{w}_{j+1}^{-1/(p-1)})^p}\\ \quad\text{if } 1<p<\infty,\\ \|\bar{w}_{j+1}^{-1}\|_{L^\infty([x,y+\varepsilon])}^{-1}+\dfrac{d}{dx}(\|\bar{w}_{j+1}^{-1}\|_{L^\infty([x,y+\varepsilon])}^{-1}) \quad\text{if } p=1\end{cases}$$

for $x\in(y,y+\varepsilon/2)$.

Remark. (1) We can define a left completion of μ with respect to y in a similar way.

(2) If $\bar{w}_{j+1}\in B_p([y,y+\varepsilon])$, then $\Lambda_p(\tilde{\mu}_j,\bar{w}_{j+1})<\infty$ for any measure $\tilde{\mu}_j$ with $\tilde{\mu}_j((y,y+\varepsilon])<\infty$. In particular, $\Lambda_p(1,\bar{w}_{j+1})<\infty$.

(3) If μ,v are two vectorial measures such that $\mu_j\geqslant cv_j$ for $0\leqslant j\leqslant k$ and $\bar{v}$ is a right completion of v, then there is a right completion $\bar{\mu}$ of μ, with $\bar{\mu}_j\geqslant c\bar{v}_j$ for $0\leqslant j\leqslant k$ (it is enough to take $\tilde{\mu}_j=\tilde{v}_j$). Also, if μ,v are comparable measures, $\bar{v}$ is a right completion of v if and only if it is comparable to a right completion $\bar{\mu}$ of μ.

(4) We always have $\bar{\mu}_k=\mu_k$ and $\bar{\mu}_j\geqslant\mu_j$ for $0\leqslant j<k$.

Definition 6. For $1\leqslant p<\infty$ and a vectorial measure μ, we say that a point $y\in\mathbf{R}$ is right j-regular (respectively, left j-regular), if there exist $\varepsilon>0$, a right completion $\bar{\mu}$ (respectively, left completion) of μ and $j<i\leqslant k$ such that $\bar{w}_i:=d\bar{\mu}_i/dx\in B_p([y,y+\varepsilon])$ (respectively, $B_p([y-\varepsilon,y])$). Also, we say that a point $y\in\mathbf{R}$ is j-regular, if it is right and left j-regular.

Remark. (1) A point $y\in\mathbf{R}$ is right j-regular (respectively, left j-regular), if at least one of the following properties is verified:

(a) There exist $\varepsilon>0$ and $j<i\leqslant k$ such that $w_i\in B_p([y,y+\varepsilon])$ (respectively, $B_p([y-\varepsilon,y])$). Here we have chosen $\tilde{\mu}_j=0$.

(b) There exist $\varepsilon>0$, $j<i\leqslant k$, $\alpha>0$, and $\delta<(i-j)p-1$, such that

$$w_i(x)\geqslant\alpha|x-y|^{\delta} \quad \text{for almost every } x\in[y,y+\varepsilon]$$

(respectively, $[y-\varepsilon,y]$). See [RARP1, Lemma 3.4].

(2) If y is right j-regular (respectively, left), then it is also right i-regular (respectively, left) for each $0\leqslant i\leqslant j$.

(3) We can take $i=j+1$ in this definition since by the second remark after Definition 5 we can choose $\bar{w}_l=w_l+1\in B_p([y,y+\varepsilon])$ for $j<l<i$, if $j+1<i$.

(4) If μ,ν are two vectorial measures with the same absolutely continuous part, then y is right j-regular (respectively, left) with respect to μ if and only if it is right j-regular (respectively, left) with respect to ν.

When we use this definition, we think of a point $\{b\}$ as the union of two half-points $\{b^+\}$ and $\{b^-\}$. With this convention, each one of the following sets:

$$(a,b)\cup(b,c)\cup\{b^+\}=(a,b)\cup[b^+,c)\neq(a,c),$$
$$(a,b)\cup(b,c)\cup\{b^-\}=(a,b^-]\cup(b,c)\neq(a,c)$$

has two connected components, and the set

$$(a,b)\cup(b,c)\cup\{b^-\}\cup\{b^+\}=(a,b)\cup(b,c)\cup\{b\}=(a,c)$$

is connected.

We only use this convention in order to study the sets of continuity of functions: we want that if $f\in C(A)$ and $f\in C(B)$, where A and B are union of intervals, then $f\in C(A\cup B)$. With the usual definition of continuity in an interval, if $f\in C([a,b))\cap C([b,c])$ then we do not have $f\in C([a,c])$. Of course, we have $f\in C([a,c])$ if and only if $f\in C([a,b^-])\cap C([b^+,c])$, where, by definition, $C([b^+,c])=C([b,c])$ and $C([a,b^-])=C([a,b])$. This idea can be formalized with a suitable topological space.

Let us introduce some notation. We denote by $\Omega^{(j)}$ the set of j-regular points or half-points, i.e., $y\in\Omega^{(j)}$ if and only if y is j-regular, we say that $y^+\in\Omega^{(j)}$ if and only if y is right j-regular, and we say that $y^-\in\Omega^{(j)}$ if and only if y is left j-regular. Obviously, $\Omega^{(k)}=\emptyset$ and $\Omega_{j+1}\cup\cdots\cup\Omega_k\subseteq\Omega^{(j)}$. Observe that $\Omega^{(j)}$ depends on p (see Definition 6).

Remark. If $0\leqslant j<k$ and I is an interval, $I\subseteq\Omega^{(j)}$, then the set $I\setminus(\Omega_{j+1}\cup\cdots\cup\Omega_k)$ is discrete (see the Remark before Definition 7 in [RARP1]).

Definition 7. We say that a function h belongs to the class $AC_{\mathrm{loc}}(\Omega^{(j)})$ if $h\in AC_{\mathrm{loc}}(I)$ for every connected component I of $\Omega^{(j)}$.

Definition 8. We say that the vectorial measure $\mu=(\mu_0,\ldots,\mu_k)$ is p-admissible if $(\mu_j)_{\mathrm{s}}(\mathbf{R}\setminus\Omega^{(j)})=0$ for $1\leqslant j\leqslant k$.

We use the letter p in p-admissible in order to emphasize the dependence on p (recall that $\Omega^{(j)}$ depends on p).

Remark. (1) There is no condition on $\mathrm{supp}(\mu_0)_s$.

(2) We have $(\mu_k)_s \equiv 0$, since $\Omega^{(k)} = \emptyset$.

(3) Every absolutely continuous measure is p-admissible.

Definition 9 (Sobolev space). Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, \ldots, \mu_k)$ a p-admissible vectorial measure. We define the Sobolev space $W^{k,p}(\Delta, \mu)$ as the space of equivalence classes of

$$V^{k,p}(\Delta, \mu) := \{f : \Delta \to \mathbf{C} / f^{(j)} \in AC_{\mathrm{loc}}(\Omega^{(j)}) \quad \text{for } 0 \leqslant j < k \text{ and}$$

$$||f^{(j)}||_{L^p(\Delta, \mu_j)} < \infty \quad \text{for } 0 \leqslant j \leqslant k\}$$

with respect to the seminorm

$$||f||_{W^{k,p}(\Delta,\mu)} := \left(\sum_{j=0}^{k} ||f^{(j)}||^p_{L^p(\Delta,\mu_j)} \right)^{1/p}.$$

Remark. (1) This definition is natural since when the $(\mu_j)_s$-measure of the set where $|f^{(j)}|$ is not continuous is positive, the integral $\int |f^{(j)}|^p d(\mu_j)_s$ does not make sense.

(2) If we consider Sobolev spaces with real valued functions every result in this paper also holds.

At this moment we can consider also norms as follows:

$$||f||^p = \int_{-1}^{1} |f|^p + \int_{-1}^{0} |x|^{p-1} |f'|^p + \int_{0}^{1} |f'|^p + |f(0^+)|^p,$$

$$||f||^p = \int_{0}^{1} |f|^p + \int_{0}^{1} |f'|^p + |f(0^+)|^p.$$

In the second example, we can write $|f(0)|^p$ instead of $|f(0^+)|^p$, since f is not defined at the left of 0, and then this causes no confusion. Obviously, we always write $(a+b)\delta_{x_0}$ instead of $a\delta_{x_0^-} + b\delta_{x_0^+}$.

Definition 10. Let us consider $1 \leqslant p < \infty$ and μ a p-admissible vectorial measure. Let us define the space $\mathscr{K}(\Delta, \mu)$ as

$$\mathscr{K}(\Delta, \mu) := \{g : \Omega^{(0)} \to \mathbf{C} / g \in V^{k,p}(\overline{\Omega^{(0)}}, \mu|_{\Omega^{(0)}}), ||g||_{W^{k,p}(\overline{\Omega^{(0)}}, \mu|_{\Omega^{(0)}})} = 0\}.$$

$\mathscr{K}(\Delta, \mu)$ is the equivalence class of 0 in $W^{k,p}(\overline{\Omega^{(0)}}, \mu|_{\Omega^{(0)}})$. It plays an important role in the general theory of Sobolev spaces and in the study of the multiplication operator in Sobolev spaces in particular (see [R2,RARP1,RARP2] and Theorems A and B).

Definition 11. Let us consider $1 \leqslant p < \infty$ and μ a p-admissible vectorial measure. We say that (Δ, μ) belongs to the class $\mathscr{C}_0$ if there exist compact sets M_n, which are a finite union of compact intervals, such that

(i) M_n intersects at most a finite number of connected components of $\Omega_1 \cup \cdots \cup \Omega_k$,
(ii) $\mathscr{K}(M_n, \mu) = \{0\}$,
(iii) $M_n \subseteq M_{n+1}$,
(iv) $\cup_n M_n = \Omega^{(0)}$.

We say that (Δ, μ) belongs to the class $\mathscr{C}$ if there exists a measure $\mu_0' = \mu_0 + \sum_{m \in D} c_m \delta_{x_m}$ with $c_m > 0$, $\{x_m\} \subset \Omega^{(0)}$, $D \subseteq \mathbf{N}$ and $(\Delta, \mu') \in \mathscr{C}_0$, where $\mu' = (\mu_0', \mu_1, \ldots, \mu_k)$ is minimal in the following sense: there exists $\{M_n\}$ corresponding to $(\Delta, \mu') \in \mathscr{C}_0$ such that if $\mu_0'' = \mu_0' - c_{m_0} \delta_{x_{m_0}}$ with $m_0 \in D$ and $\mu'' = (\mu_0'', \mu_1, \ldots, \mu_k)$, then $\mathscr{K}(M_n, \mu'') \neq \{0\}$ if $x_{m_0} \in M_n$.

Remark. (1) Condition $(\Delta, \mu) \in \mathscr{C}$ is not very restrictive. In fact, the proof of Theorem A (see [RARP1, Theorem 4.3]) gives that if $\Omega^{(0)} \setminus (\Omega_1 \cup \ldots \cup \Omega_k)$ has only a finite number of points in each connected component of $\Omega^{(0)}$, then $(\Delta, \mu) \in \mathscr{C}$. If furthermore $\mathscr{K}(\Delta, \mu) = \{0\}$, we have $(\Delta, \mu) \in \mathscr{C}_0$.

(2) The proof of Theorem A gives that if for every connected component Λ of $\Omega_1 \cup \cdots \cup \Omega_k$ we have $\mathscr{K}(\bar{\Lambda}, \mu) = \{0\}$, then $(\Delta, \mu) \in \mathscr{C}_0$. Condition $\#\mathrm{supp}\, \mu_0|_{\bar{\Lambda} \cap \Omega^{(0)}} \geqslant k$ implies $\mathscr{K}(\bar{\Lambda}, \mu) = \{0\}$.

(3) Since the restriction of a function of $\mathscr{K}(\Delta, \mu)$ to M_n is in $\mathscr{K}(M_n, \mu)$ for every n, then $(\Delta, \mu) \in \mathscr{C}_0$ implies $\mathscr{K}(\Delta, \mu) = \{0\}$.

(4) If $(\Delta, \mu) \in \mathscr{C}_0$, then $(\Delta, \mu) \in \mathscr{C}$, with $\mu' = \mu$.

The next results, proved in [RARP1], play a central role in the theory of Sobolev spaces with respect to measures (see the proofs in [RARP1, Theorems 4.3 and 5.1]).

Theorem A. *Let us suppose that* $1 \leqslant p < \infty$ and $\mu = (\mu_0, \ldots, \mu_k)$ *is a* p*-admissible vectorial measure. Let* K_j *be a finite union of compact intervals contained in* $\Omega^{(j)}$, *for* $0 \leqslant j < k$ *and* $\bar{\mu}$ *a right* (*or left*) *completion of* μ. *Then*:

(a) *If* $(\Delta, \mu) \in \mathscr{C}_0$ *there exist positive constants* $c_1 = c_1(K_0, \ldots, K_{k-1})$ *and* $c_2 = c_2(\bar{\mu}, K_0, \ldots, K_{k-1})$ *such that*

$$c_1 \sum_{j=0}^{k-1} ||g^{(j)}||_{L^\infty(K_j)} \leqslant ||g||_{W^{k,p}(\Delta,\mu)},$$

$$c_2 ||g||_{W^{k,p}(\Delta,\bar{\mu})} \leqslant ||g||_{W^{k,p}(\Delta,\mu)} \quad \forall g \in V^{k,p}(\Delta, \mu).$$

(b) *If* $(\Delta, \mu) \in \mathscr{C}$ *there exist positive constants* $c_3 = c_3(K_0, \ldots, K_{k-1})$ *and* $c_4 = c_4(\bar{\mu}, K_0, \ldots, K_{k-1})$ *such that for every* $g \in V^{k,p}(\Delta, \mu)$, *there exists* $g_0 \in V^{k,p}(\Delta, \mu)$,

independent of $K_0, \ldots, K_{k-1}$, c_3, c_4 *and* $\bar{\mu}$, *with*

$$||g_0 - g||_{W^{k,p}(\Delta,\mu)} = 0,$$

$$c_3 \sum_{j=0}^{k-1} ||g_0^{(j)}||_{L^\infty(K_j)} \leqslant ||g_0||_{W^{k,p}(\Delta,\mu)} = ||g||_{W^{k,p}(\Delta,\mu)},$$

$$c_4 ||g_0||_{W^{k,p}(\Delta,\bar{\mu})} \leqslant ||g||_{W^{k,p}(\Delta,\mu)}.$$

Furthermore, if g_0, f_0 *are these representatives of* g, f, *respectively, we have for the same constants* c_3, c_4

$$c_3 \sum_{j=0}^{k-1} ||g_0^{(j)} - f_0^{(j)}||_{L^\infty(K_j)} \leqslant ||g - f||_{W^{k,p}(\Delta,\mu)},$$

$$c_4 ||g_0 - f_0||_{W^{k,p}(\Delta,\bar{\mu})} \leqslant ||g - f||_{W^{k,p}(\Delta,\mu)}.$$

Remark. Theorem A is proved in [RARP1] with the additional hypothesis that $\tilde{\mu} := \bar{\mu} - \mu$ is absolutely continuous, since [RARP1] only uses absolutely continuous completions, but the same proof also works in the general case.

Theorem B. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a p-admissible vectorial measure with* $(\Delta, \mu) \in \mathscr{C}$. *Then the Sobolev space* $W^{k,p}(\Delta, \mu)$ *is complete.*

A result on density of smooth functions is the following. It is a particular case of Theorem 4.1 in [RARP2]. We do not write the complete statement of Theorem 4.1 in [RARP2] since we would need several definitions.

Theorem C. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a finite p-admissible vectorial measure with* $\Delta = [a, b]$. *If* $w_k := d\mu_k/dx \in B_p([a, b])$, *then* $C_c^\infty(\mathbf{R})$ *is dense in the Sobolev space* $W^{k,p}([a, b], \mu)$.

Remark. Under the hypotheses of Theorem C, μ is p-admissible if and only if $(\mu_k)_s = 0$, since $\Omega^{(k-1)} = [a, b]$.

We need more results appearing in [R2,RARP1,RARP2]. An immediate modification of Lemma 3.3 in [RARP1] gives the following proposition.

Proposition A. *Let* $1 \leqslant p < \infty$ *and let* $\mu = (\mu_0, \ldots, \mu_k)$ *be a p-admissible vectorial measure in* $[a, b]$, *with* $w_{k_0} := d\mu_{k_0}/dx \in B_p((a, b])$ *for some* $0 < k_0 \leqslant k$. *If we construct a right completion* $\bar{\mu}$ *of* μ *with respect to the point a taking* $\varepsilon = b - a$, *and* $\bar{\mu}_j = \mu_j$ *for* $k_0 \leqslant j \leqslant k$, *then there exist positive constants* c_j *such that*

$$c_j ||g^{(j)}||_{L^p([a,b],\bar{\mu}_j)} \leqslant \sum_{i=j}^{k_0} ||g^{(i)}||_{L^p([a,b],\mu_i)} + \sum_{i=j}^{k_0-1} |g^{(i)}(b)|,$$

for all $0 \leqslant j < k_0$ *and* $g \in V^{k,p}([a,b],\mu)$. *In particular, there is a positive constant* c *such that*

$$c||g||_{W^{k,p}([a,b],\tilde{\mu})} \leqslant ||g||_{W^{k,p}([a,b],\mu)} + \sum_{j=0}^{k_0-1} |g^{(j)}(b)|,$$

for all $g \in V^{k,p}([a,b],\mu)$.

Corollary 4.3 in [RARP1] gives the following result.

Corollary A. *Let us suppose that* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *is a p-admissible vectorial measure. Let* K_j *be a finite union of compact intervals contained in* $\Omega^{(j)}$, *for* $0 \leqslant j < k$. *Then*:

(a) *If* $(\Delta,\mu) \in \mathscr{C}_0$ *there exists a positive constant* $c_1 = c_1(K_0, \ldots, K_{k-1})$ *such that*

$$c_1 \sum_{j=0}^{k-1} ||g^{(j+1)}||_{L^1(K_j)} \leqslant ||g||_{W^{k,p}(\Delta,\mu)} \quad \forall g \in V^{k,p}(\Delta,\mu).$$

(b) *If* $(\Delta,\mu) \in \mathscr{C}$ *there exists a positive constant* $c_2 = c_2(K_0, \ldots, K_{k-1})$ *such that for every* $g \in V^{k,p}(\Delta,\mu)$, *there exists* $g_0 \in V^{k,p}(\Delta,\mu)$ (*the same function as in Theorem A*), *with*

$$||g_0 - g||_{W^{k,p}(\Delta,\mu)} = 0,$$

$$c_2 \sum_{j=0}^{k-1} ||g_0^{(j+1)}||_{L^1(K_j)} \leqslant ||g_0||_{W^{k,p}(\Delta,\mu)} = ||g||_{W^{k,p}(\Delta,\mu)}.$$

Furthermore, if g_0, f_0 *are these representatives of* g, f, *respectively, we have for the same constant* c_2

$$c_2 \sum_{j=0}^{k-1} ||g_0^{(j+1)} - f_0^{(j+1)}||_{L^1(K_j)} \leqslant ||g - f||_{W^{k,p}(\Delta,\mu)}.$$

A simple modification in the proof of Corollary A gives Corollary B. Recall that we use the symbol $W^{k-m,p}(\Delta,\mu)$ to denote the Sobolev space $W^{k-m,p}(\Delta,(\mu_m, \ldots, \mu_k))$.

Corollary B. *Let us suppose that* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *is a p-admissible vectorial measure. For some* $0 < m \leqslant k$, *assume that* $(\Delta, (\mu_m, \ldots, \mu_k)) \in \mathscr{C}_0$. *Let* K *be a finite union of compact intervals contained in* $\Omega^{(m-1)}$. *Then there exists a positive constant* $c_1 = c_1(K)$ *such that*

$$c_1 ||g||_{L^1(K)} \leqslant ||g||_{W^{k-m,p}(\Delta,\mu)} \quad \forall g \in V^{k-m,p}(\Delta,\mu).$$

An immediate computation gives the following result.

Lemma A. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a p-admissible vectorial measure with*

$$d\mu_{j+1}(x) \leqslant c_1^p |x - a_0|^p d\mu_j(x)$$

for $0 \leqslant j < k$, $a_0 \in \mathbf{R}$ *and* x *in an interval* I. *Let* $\varphi \in C^k(\mathbf{R})$ *be such that* $\operatorname{supp} \varphi' \subseteq [\lambda, \lambda + t]$, *with* $\max\{|\lambda - a_0|, |\lambda + t - a_0|\} \leqslant c_2 t$ *and* $||\varphi^{(j)}||_{L^\infty(I)} \leqslant c_3 t^{-j}$ *for* $0 \leqslant j \leqslant k$. *Then, there is a positive constant* c_0 *which is independent of* $I, a_0, \lambda, t, \mu, \varphi$ *and* g *such that*

$$||\varphi g||_{W^{k,p}(\Delta,\mu)} \leqslant c_0 ||g||_{W^{k,p}(I,\mu)},$$

for every $g \in W^{k,p}(\Delta, \mu)$ *with* $\operatorname{supp}(\varphi g) \subseteq I$.

The next result is a consequence of Corollary 3.2 in [R2].

Corollary C. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a locally finite p-admissible vectorial measure. Assume that there exist* $a_0 \in \mathbf{R}$, *an integer* $0 \leqslant r < k$, *an open neighbourhood* U *of* a_0 *and* $c > 0$ *such that*

$$d\mu_{j+1}(x) \leqslant c|x - a_0|^p d\mu_j(x)$$

for $x \in U$ *and* $r \leqslant j < k$. *Then* a_0 *is neither right nor left r-regular*.

The following theorems [RARP2, 4.2 and 4.3] give density results for measures which can be obtained by "gluing" simpler ones.

Theorem D. *Let us consider* $1 \leqslant p < \infty$, $-\infty \leqslant a < b < c < d \leqslant \infty$. *Let* $\mu = (\mu_0, \ldots, \mu_k)$ *be a p-admissible vectorial measure in* $[a, d]$, *and assume that there exists an interval* $I \subseteq [b, c]$ *with* $(I, \mu) \in \mathscr{C}_0$ *and* $\mu_j(I) < \infty$ *for* $0 \leqslant j \leqslant k$. *Then* $C^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([a, d], \mu)$ *if and only if* $C^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([a, c], \mu)$ *and* $W^{k,p}([b, d], \mu)$.

Theorem E. *Let us consider* $1 \leqslant p < \infty$ *and* $\{a_n\}$, $\{b_n\}$ *strictly increasing sequences of real numbers* (n *belonging to a finite set, to* $\mathbf{Z}$, $\mathbf{Z}^+$ *or* $\mathbf{Z}^-$) *with* $a_{n+1} < b_n$ *for every* n. *Let* $\mu = (\mu_0, \ldots, \mu_k)$ *be a p-admissible vectorial measure in* $(\alpha, \beta) := \cup_n (a_n, b_n)$, *with* $-\infty \leqslant \alpha < \beta \leqslant \infty$. *Assume that for each* n *there exists an interval* $I_n \subseteq [a_{n+1}, b_n]$ *with* $(I_n, \mu) \in \mathscr{C}_0$ *and* $\mu_j(I_n) < \infty$ *for* $0 \leqslant j \leqslant k$. *Then* $C^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([\alpha, \beta], \mu)$ *if and only if* $C^\infty(\mathbf{R})$ *is dense in every* $W^{k,p}([a_n, b_n], \mu)$.

3. The case of compact support

Recall that under the hypothesis Δ compact, it is equivalent to prove the density in $W^{k,p}(\Delta, \mu)$ of $C_c^\infty(\mathbf{R})$, $C^\infty(\mathbf{R})$ or P.

Lemma 3.1. *Let us consider* $1 \leqslant p_1, \ldots, p_m < \infty$ *and measures* $\mu_1, \ldots, \mu_m$ *in a measurable space* X. *Let us assume that* (X, μ) *satisfies the conclusion of Lusin*

Theorem, where $\mu := \mu_1 + \cdots + \mu_m$. *Then any function in* $L^{p_1}(X,\mu_1) \cap \cdots \cap L^{p_m}(X,\mu_m)$ *can be approximated in the norm*

$$||\cdot||_X := ||\cdot||_{L^{p_1}(X,\mu_1)} + \cdots + ||\cdot||_{L^{p_m}(X,\mu_m)}$$

by functions in $C_c(X)$.

Remark. Recall that any finite measure in $\mathbf{R}$ satisfies the conclusion of Lusin Theorem.

Proof. We denote by S the set of simple and measurable functions s on X with complex values such that

$$\mu(\{x \in X : s(x) \neq 0\}) < \infty.$$

We prove now that any function in $L^{p_1}(X,\mu_1) \cap \cdots \cap L^{p_m}(X,\mu_m)$ can be approximated in the norm $||\cdot||_X$ by functions in S. Let us consider a function f in $L^{p_1}(X,\mu_1) \cap \cdots \cap L^{p_m}(X,\mu_m)$. Without loss of generality, we can assume that $f \geqslant 0$. As usual, for $n \in \mathbf{N}$ and $1 \leqslant i \leqslant n2^n$ we define

$$E_{n,i} := f^{-1}([(i-1)2^{-n}, i2^{-n})), \quad F_n := f^{-1}([n,\infty]),$$

$$s_n := \sum_{i=1}^{n2^n} (i-1)2^{-n}\chi_{E_{n,i}} + n\chi_{F_n}.$$

Obviously we have $0 \leqslant s_n \leqslant f$, $||s_n||_X < \infty$ and $s_n \in S$. The conclusion $\lim_{n\to\infty} ||f - s_n||_X = 0$ is immediate by Dominated Convergence Theorem, since $|f - s_n|^{p_j} \leqslant f^{p_j}$ for $1 \leqslant j \leqslant m$.

Therefore, it is enough to prove the lemma for functions in S. Take $s \in S$ and $\varepsilon > 0$. By hypothesis there exists $g \in C_c(X)$ such that $|g(x)| \leqslant ||s||_{L^\infty(X,\mu)}$ for every $x \in X$, and $g(x) = s(x)$ except for a set of μ-measure less than ε. Therefore

$$||g - s||_{L^{p_j}(X,\mu_j)} \leqslant 2\varepsilon^{1/p_j}||s||_{L^\infty(X,\mu)}$$

for each $1 \leqslant j \leqslant m$. This finishes the proof. □

Lemma 3.2. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a finite p-admissible vectorial measure with* $\Delta \subseteq [a,b]$. *Then any function in* $W^{k,p}(\Delta,\mu) \cap W^{k,1}([a,b])$ *can be approximated by functions in* $C_c^\infty(\mathbf{R})$ *with the norm* $W^{k,p}(\Delta,\mu)$.

Proof. Take any fixed function f in $W^{k,p}(\Delta,\mu) \cap W^{k,1}([a,b])$. Since μ_k is finite, Lemma 3.1 gives that for each $\varepsilon > 0$ we can find a function $h_0 \in C_c((a,b))$ with

$$||f^{(k)} - h_0||_{L^p(\Delta,w_k)} < \varepsilon, \quad ||f^{(k)} - h_0||_{L^1([a,b])} < \varepsilon.$$

By convolution with an approximation of identity, we can find a function $h \in C_c^\infty((a,b))$ with

$$||h - h_0||_{L^\infty([a,b])} < \varepsilon, \quad ||h - h_0||_{L^1([a,b])} < \varepsilon.$$

Therefore, we have that

$$||h-h_0||_{L^p([a,b],w_k)}<\mu_k([a,b])^{1/p}\varepsilon,$$

and then

$$||f^{(k)}-h||_{L^p([a,b],w_k)}<c\varepsilon, \quad ||f^{(k)}-h||_{L^1([a,b])}<2\varepsilon.$$

Let us fix $x_0\in(a,b)$. The function

$$g(x):=f(x_0)+\cdots+f^{(k-1)}(x_0)\frac{(x-x_0)^{k-1}}{(k-1)!}+\int_{x_0}^{x}h(t)\frac{(x-t)^{k-1}}{(k-1)!}\,dt,$$

belongs to $C^\infty(\mathbf{R})$ and we have

$$g^{(j)}(x):=f^{(j)}(x_0)+\cdots+f^{(k-1)}(x_0)\frac{(x-x_0)^{k-j-1}}{(k-j-1)!}+\int_{x_0}^{x}h(t)\frac{(x-t)^{k-j-1}}{(k-j-1)!}\,dt$$

for $0\leqslant j<k$. Therefore

$$\begin{aligned}||f^{(j)}-g^{(j)}||_{L^p([a,b],\mu_j)}&\leqslant\left(\int_a^b\left(\int_a^b|f^{(k)}(t)-h(t)|\frac{|x-t|^{k-j-1}}{(k-j-1)!}\,dt\right)^p d\mu_j(x)\right)^{1/p}\\&\leqslant c||f^{(k)}-h||_{L^1([a,b])}<c\varepsilon,\end{aligned}$$

for $0\leqslant j<k$, since $\mu_j([a,b])<\infty$. Hence, we have obtained

$$||f-g||_{W^{k,p}(\Delta,\mu)}<c\varepsilon$$

with $g\in C^\infty(\mathbf{R})$. □

The following results are improvements of Theorem 4.1 in [RARP2].

Theorem 3.1. *Let us consider $1\leqslant p<\infty$ and $\mu=(\mu_0,\mu_1)$ a finite p-admissible vectorial measure with $\Delta=[a,b]$ and $w_1:=d\mu_1/dx\in B_p((a,b))$. Then $C_c^\infty(\mathbf{R})$ is dense in $W^{1,p}([a,b],\mu)$.*

Proof. Assume first that $w_k\in B_p((a,b])$. If $w_1\in B_p([a,b])$, the theorem is a consequence of Theorem C. So we can assume that $w_1\in B_p((a,b])\setminus B_p([a,b])$; this implies that a is not right 0-regular.

By Lemma 3.2 it is enough to show that any function f in $V^{1,p}([a,b],\mu)$ can be approximated by functions f_n in $V^{1,p}([a,b],\mu)\cap W^{1,1}([a,b])$, i.e. by functions $f_n\in V^{1,p}([a,b],\mu)$ with $f_n'\in L^1([a,b])$.

If $f\in V^{1,p}([a,b],\mu)$, then $f\in AC_{\mathrm{loc}}((a,b])$, since $w_1\in B_p((a,b])$. Let us choose $0<t_n\leqslant 1/n$ such that

$$\tfrac{1}{2}|f(a+t_n)|\leqslant|f(x)| \quad \text{for every } x\in(a,a+1/n]. \tag{3.1}$$

Define the functions $g_n := f'\chi_{[a+t_n,b]}$ and

$$f_n(x) := f(b) + \int_b^x g_n.$$

Observe that Hölder inequality gives that $f_n' = g_n \in L^1([a,b])$:

$$\begin{aligned} ||g_n||_{L^1([a,b])} &= \int_{a+t_n}^b |f'| w_1^{1/p} w_1^{-1/p} \\ &\leqslant \left(\int_{a+t_n}^b |f'|^p w_1 \right)^{1/p} ||w_1^{-1}||^{1/p}_{L^{1/(p-1)}([a+t_n,b])} < \infty, \end{aligned}$$

since we have $w_1 \in B_p((a,b])$ and $f' \in L^p([a,b], w_1)$. It is clear that

$$\int_a^b |f' - f_n'|^p w_1 = \int_a^b |f' - g_n|^p w_1 = \int_a^{a+t_n} |f'|^p w_1 \to 0$$

as $n \to \infty$. Observe that $\mu_0(\{a\}) = 0$, since a is not right 0-regular and μ is p-admissible. This fact and (3.1) give

$$\begin{aligned} \int_a^b |f(x) - f_n(x)|^p d\mu_0(x) &= \int_a^{a+t_n} |f(x) - f(a+t_n)|^p \, d\mu_0(x) \\ &\leqslant 3^p \int_a^{a+t_n} |f(x)|^p \, d\mu_0(x) \to 0 \end{aligned}$$

as $n \to \infty$. Hence, the proof is finished in this case, since f_n converges to f in $W^{1,p}([a,b], \mu)$ and $f_n' = g_n \in L^1([a,b])$.

A symmetric argument gives the case $w_k \in B_p([a,b))$. The general case is easy now. Choose a compact interval $I = [\alpha, \beta] \subset (a,b) \subseteq \Omega^{(0)}$. Then $V^{1,p}([a,b],\mu) \subset C(I)$ and $V^{1,p}([a,b],\mu) = V^{1,p}([a,b],\tilde{\mu})$, with $\tilde{\mu} = (\tilde{\mu}_0, \mu_1)$ and $d\tilde{\mu}_0 = d\mu_0 + \chi_I \, dx$. We have that $\mathscr{K}(I,\tilde{\mu}) = \{0\}$ (see Remark 2 to Definition 11) and even $(I,\tilde{\mu}) \in \mathscr{C}_0$, since $\Omega^{(0)} \setminus \Omega_1$ restricted to I is the empty set (see Remark 1 to Definition 11). We have proved that $C_c^\infty(\mathbf{R})$ is dense in $W^{1,p}([a,\beta],\tilde{\mu})$ and $W^{1,p}([\alpha,b],\tilde{\mu})$. Since $(I,\tilde{\mu}) \in \mathscr{C}_0$, Theorem D gives that $C_c^\infty(\mathbf{R})$ is dense in $W^{1,p}([a,b],\tilde{\mu})$ and therefore in $W^{1,p}([a,b],\mu)$. □

Theorem 3.2. *Let us consider* $1 \leqslant p < \infty$, $0 < m \leqslant k$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a finite p-admissible vectorial measure with* $\Delta = [a,b]$ *and* $C_c^\infty(\mathbf{R})$ *dense in* $W^{k-m,p}([a,b],\mu)$. *Assume that* $([a,b], (\mu_m, \ldots, \mu_k)) \in \mathscr{C}_0$ *if* $m < k$. *Assume also that we have either*:

(1) $\Omega^{(m-1)} = [a,b]$,
(2) $\Omega^{(m-1)} = (a,b]$ *and there exists* $\varepsilon > 0$ *such that* $\mu|_{[a,a+\varepsilon]}$ *is a right completion of the vectorial measure* $(0, \ldots, 0, \mu_m, \ldots, \mu_k)$.

Then $C_c^\infty(\mathbf{R})$ *is dense in* $W^{k,p}([a,b],\mu)$.

Remark. The same conclusion is true if we change (2) by (2′): $\Omega^{(m-1)} = [a,b)$ and there exists $\varepsilon > 0$ such that $\mu|_{[b-\varepsilon,b]}$ is a left completion of the vectorial measure $(0, \ldots, 0, \mu_m, \ldots, \mu_k)$.

Proof. Consider case (1). If $m=k$, then $\Omega^{(k-1)}=[a,b]$ gives $w_k\in B_p([a,b])$ and Theorem C gives the result. Assume now $0<m<k$. Given a function $f\in V^{k,p}([a,b],\mu)$, we can choose a sequence $\{g_n\}\subset C_c^\infty(\mathbf{R})$ converging to $f^{(m)}$ in $W^{k-m,p}([a,b],\mu)$. Define

$$f_n(x):=f(b)+\cdots+f^{(m-1)}(b)\frac{(x-b)^{m-1}}{(m-1)!}+\int_b^x g_n(t)\frac{(x-t)^{m-1}}{(m-1)!}\,dt.$$

It is clear that

$$f_n^{(j)}(x)=f^{(j)}(b)+\cdots+f^{(m-1)}(b)\frac{(x-b)^{m-j-1}}{(m-j-1)!}+\int_b^x g_n(t)\frac{(x-t)^{m-j-1}}{(m-j-1)!}\,dt \tag{3.2}$$

for $0\leqslant j\leqslant m-1$. Since $([a,b],(\mu_m,\ldots,\mu_k))\in\mathscr{C}_0$, Corollary B gives

$$||f^{(m)}-f_n^{(m)}||_{L^1([a,b])}\leqslant c||f^{(m)}-f_n^{(m)}||_{W^{k-m,p}([a,b],\mu)}.$$

It is immediate by (3.2) that

$$||f^{(j)}-f_n^{(j)}||_{L^p([a,b],\mu_j)}\leqslant c||f^{(m)}-f_n^{(m)}||_{L^1([a,b])}\leqslant c||f^{(m)}-f_n^{(m)}||_{W^{k-m,p}([a,b],\mu)}$$

for $0\leqslant j\leqslant m-1$, since μ_j is finite, and we conclude that f_n converges to f in $W^{k,p}([a,b],\mu)$.

Consider now case (2). Given a function $f\in V^{k,p}([a,b],\mu)$, let us consider a sequence $\{g_n\}\subset C_c^\infty(\mathbf{R})$ converging to $f^{(m)}$ in $W^{k-m,p}([a,b],\mu)$. Define the functions h_n by

$$h_n(x):=f(a+\varepsilon)+\cdots+f^{(m-1)}(a+\varepsilon)\frac{(x-a-\varepsilon)^{m-1}}{(m-1)!}+\int_{a+\varepsilon}^x g_n(t)\frac{(x-t)^{m-1}}{(m-1)!}\,dt.$$

We have

$$h_n^{(j)}(x)=f^{(j)}(a+\varepsilon)+\cdots+f^{(m-1)}(a+\varepsilon)\frac{(x-a-\varepsilon)^{m-j-1}}{(m-j-1)!}+\int_{a+\varepsilon}^x g_n(t)\frac{(x-t)^{m-j-1}}{(m-j-1)!}\,dt$$

for $0\leqslant j\leqslant m-1$.

If $m<k$, since $([a,b],(\mu_m,\ldots,\mu_k))\in\mathscr{C}_0$, Corollary B gives

$$||f^{(m)}-h_n^{(m)}||_{L^1([a+\varepsilon,b])}\leqslant c||f^{(m)}-h_n^{(m)}||_{W^{k-m,p}([a,b],\mu)}.$$

This inequality is also true for $m=k$, since then $w_k \in B_p((a,b])$. It is immediate that

$$\begin{aligned}||f^{(j)}-h_n^{(j)}||_{L^p([a+\varepsilon,b],\mu_j)} &\leqslant c||f^{(m)}-h_n^{(m)}||_{L^1([a+\varepsilon,b])}\\ &\leqslant c||f^{(m)}-h_n^{(m)}||_{W^{k-m,p}([a,b],\mu)},\end{aligned}$$

for $0\leqslant j\leqslant m-1$. This gives

$$||f-h_n||_{W^{k,p}([a+\varepsilon,b],\mu)}\leqslant c||f^{(m)}-h_n^{(m)}||_{W^{k-m,p}([a,b],\mu)}. \tag{3.3}$$

Proposition A gives that there are a positive constant c and $0<k_0\leqslant k$ such that

$$c||g||_{W^{k,p}([a,a+\varepsilon],\mu)}\leqslant ||g||_{W^{k,p}([a,a+\varepsilon],(0,\ldots,0,\mu_m,\ldots,\mu_k))}+\sum_{j=0}^{k_0-1}|g^{(j)}(a+\varepsilon)|$$

for all $g\in V^{k,p}([a,b],\mu)$, where $a+\varepsilon$ is left (k_0-1)-regular. If $k_0>m$, since $([a,b],(\mu_m,\ldots,\mu_k))\in\mathscr{C}_0$, Theorem A gives

$$\sum_{j=m}^{k_0-1}|g^{(j)}(a+\varepsilon)|\leqslant c||g^{(m)}||_{W^{k-m,p}([a,b],\mu)}$$

for all $g\in V^{k,p}([a,b],\mu)$. Consequently, we have for $0<k_0\leqslant k$,

$$c||g||_{W^{k,p}([a,a+\varepsilon],\mu)}\leqslant ||g^{(m)}||_{W^{k-m,p}([a,b],\mu)}+\sum_{j=0}^{m-1}|g^{(j)}(a+\varepsilon)|$$

for all $g\in V^{k,p}([a,b],\mu)$. Since $(f-h_n)^{(j)}(a+\varepsilon)=0$, for $0\leqslant j\leqslant m-1$, we have

$$||f-h_n||_{W^{k,p}([a,a+\varepsilon],\mu)}\leqslant c||f^{(m)}-h_n^{(m)}||_{W^{k-m,p}([a,b],\mu)}.$$

This inequality and (3.3) imply that h_n converges to f in $W^{k,p}([a,b],\mu)$. □

Theorem 3.3. *Let us consider $1\leqslant p<\infty$ and $\mu=(\mu_0,\ldots,\mu_k)$ a finite p-admissible vectorial measure with $\Delta=[a,b]$ and $w_k:=d\mu_k/dx\in B_p((a,b])$. Assume that we have either*:

(1) *a is right $(k-2)$-regular if $k\geqslant 2$,*
(2) *there exists $\varepsilon>0$ such that $\mu|_{[a,a+\varepsilon]}$ is a right completion of $(0,\ldots,0,\mu_{k-1},\mu_k)$ if $k\geqslant 2$.*

Then $C_c^\infty(\mathbf{R})$ is dense in $W^{k,p}([a,b],\mu)$.

Remark. The same conclusion is true if we change (2) by (2′): there exists $\varepsilon>0$ such that $\mu|_{[b-\varepsilon,b]}$ is a left completion of the vectorial measure $(0,\ldots,0,\mu_{k-1},\mu_k)$ if $k\geqslant 2$.

Proof. If $k=1$ the result is Theorem 3.1. Therefore we can assume $k\geqslant 2$. Choose a compact interval $I\subset(a,b]\subseteq\Omega^{(k-1)}$; then every function $u\in V^{k,p}([a,b],\mu)$ verifies $u^{(k-1)}\in C(I)$ and then u belongs to $V^{k,p}([a,b],\tilde{\mu})$ with $\tilde{\mu}=(\mu_0,\ldots,\mu_{k-2},\tilde{\mu}_{k-1},\mu_k)$ and $d\tilde{\mu}_{k-1}=d\mu_{k-1}+\chi_I dx$. We have that $\mathscr{K}([a,b],(\tilde{\mu}_{k-1},\mu_k))=\{0\}$ (see Remark 2 to

Definition 11) and even $([a,b],(\tilde{\mu}_{k-1},\mu_k))\in\mathscr{C}_0$, since $\Omega^{(k-1)}\setminus\Omega_k\subseteq\{a,b\}$ (see Remark 1 to Definition 11).

Theorem 3.1 gives that $C_c^\infty(\mathbf{R})$ is dense in $W^{1,p}([a,b],\tilde{\mu})$. Consequently, the measure $\tilde{\mu}$ satisfies the hypotheses in Theorem 3.2 with $m=k-1$, and we have that $C_c^\infty(\mathbf{R})$ is dense in $W^{k,p}([a,b],\tilde{\mu})$ and therefore in $W^{k,p}([a,b],\mu)$. □

Theorem 3.4. *Let us consider $1\leqslant p<\infty$, a compact interval I and a finite p-admissible vectorial measure $\mu=(\mu_0,\ldots,\mu_k)$ with $\Delta=I$. Assume that there exist $a_0\in I$, an integer $0\leqslant r<k$ and positive constants c,δ such that*

(1) $d\mu_{j+1}(x)\leqslant c|x-a_0|^p d\mu_j(x)$ *in* $[a_0-\delta,a_0+\delta]\cap I$ *for* $r\leqslant j<k$,
(2) $w_k:=d\mu_k/dx\in B_p(I\setminus\{a_0\})$,
(3) *if* $r>0$, a_0 *is* $(r-1)$*-regular.*

Then $C_c^\infty(\mathbf{R})$ is dense in $W^{k,p}(I,\mu)$.

Remark. Condition (1) means that μ_{j+1} is absolutely continuous with respect to μ_j in $[a_0-\delta,a_0+\delta]\cap I$ and its Radon–Nikodym derivative $d\mu_{j+1}/d\mu_j$ is less or equal than $c|x-a_0|^p$. Proposition 3.2 in [R2] shows that this condition is not as restrictive as it seems, since many vectorial weights with analytic singularities can be modified in order to satisfy (1).

Proof. Without loss of generality, we can assume that a_0 is an interior point of I, since the argument is simpler if $a_0\in\partial I$. Let us take $f\in V^{k,p}(I,\mu)$. Consider now a function $\varphi\in C_c^\infty(\mathbf{R})$ with $\varphi=1$ in $[-1,1]$, $\varphi=0$ in $\mathbf{R}\setminus(-2,2)$, and $0\leqslant\varphi\leqslant1$ in $\mathbf{R}$. For each $n\in\mathbf{N}$, let us define $\varphi_n(x):=\varphi(n(x-a_0))$ and $h_n:=(1-\varphi_n)f^{(r)}$. We have

$$||f^{(r)}-h_n||_{W^{k-r,p}(I,\mu)}=||\varphi_n f^{(r)}||_{W^{k-r,p}(I,\mu)}\leqslant c_0||f^{(r)}||_{W^{k-r,p}([a_0-2/n,a_0+2/n],\mu)},$$

since we are in the hypothesis of Lemma A, with $\lambda=a_0-2/n$, $t=4/n$ and $I=[a_0-2/n,a_0+2/n]$: observe that $|\lambda-a_0|=|\lambda+t-a_0|=2/n=t/2$ and

$$\begin{aligned}||\varphi_n^{(j)}||_{L^\infty(\mathbf{R})}&=n^j||\varphi^{(j)}||_{L^\infty(\mathbf{R})}\\&\leqslant 4^k\max\{||\varphi||_{L^\infty(\mathbf{R})},||\varphi'||_{L^\infty(\mathbf{R})},\ldots,||\varphi^{(k)}||_{L^\infty(\mathbf{R})}\}t^{-j}.\end{aligned}$$

Corollary C gives that a_0 is neither right nor left r-regular; this fact implies $\mu_r(\{a_0\})=\cdots=\mu_k(\{a_0\})=0$. Hence, we deduce that $||f^{(r)}-h_n||_{W^{k-r,p}(I,\mu)}\to0$ as $n\to\infty$. Define $\mu^n:=(\mu_0,\ldots,\mu_{k-1},\mu_k^n)$, with $d\mu_k^n:=d\mu_k+\chi_{[a_0-1/n,a_0+1/n]}dx$; hypothesis (2) gives that $d\mu_k^n/dx=w_k+\chi_{[a_0-1/n,a_0+1/n]}\in B_p(I)$. Then Theorem C gives that each function h_n can be approximated by functions in $C_c^\infty(\mathbf{R})$ with respect to the norm $W^{k-r,p}(I,\mu^n)$ (since I is compact and μ is finite) and therefore in $W^{k-r,p}(I,\mu)$. This finishes the proof if $r=0$. Otherwise, hypothesis (2) and (3) give $\Omega^{(r-1)}=I$ and consequently $f^{(r-1)}\in AC(I)$.

Without loss of generality, we can assume that there exists $\varepsilon>0$ such that $[a_0-\varepsilon,a_0+\varepsilon]$ is contained in the interior of I and $w_r\geqslant1$ in $I\setminus[a_0-\varepsilon,a_0+\varepsilon]$. Otherwise,

we can change μ by μ^* with $\mu_j^* := \mu_j$ if $j \neq r$ and $d\mu_r^* := d\mu_r + \chi_{I \setminus [a_0-\varepsilon, a_0+\varepsilon]}\, dx$. It is obvious that it is more complicated to approximate f in $W^{k,p}(I, \mu^*)$ than in $W^{k,p}(I, \mu)$. Therefore, we have $\mathscr{K}([a,b], (\mu_r, \ldots, \mu_k)) = \{0\}$ (see Remark 2 to Definition 11) and even $([a,b], (\mu_r, \ldots, \mu_k)) \in \mathscr{C}_0$, since $(a,b)^{(r)} \setminus ((a,b)_{r+1} \cup \cdots \cup (a,b)_k)$ has at most three points (see Remark 1 to Definition 11).

Let us consider a sequence $\{q_n\} \subset C_c^\infty(\mathbf{R})$ converging to $f^{(r)}$ in $W^{k-r,p}(I, \mu)$. Corollary B gives

$$||f^{(r)} - q_n||_{L^1(I)} \leqslant c||f^{(r)} - q_n||_{W^{k-r,p}(I,\mu)}.$$

For any fixed $a \in I$, the smooth functions defined by

$$Q_n(x) := f(a) + f'(a)(x-a) + \cdots + f^{(r-1)}(a)\frac{(x-a)^{r-1}}{(r-1)!} + \int_a^x q_n(t)\frac{(x-t)^{r-1}}{(r-1)!}\,dt$$

satisfy

$$||f^{(j)} - Q_n^{(j)}||_{L^p(I,\mu_j)} \leqslant c||f^{(r)} - q_n||_{L^1(I)}$$

for $0 \leqslant j < r$, and consequently

$$\begin{aligned} ||f - Q_n||_{W^{k,p}(I,\mu)} &\leqslant c||f^{(r)} - q_n||_{L^1(I)} + ||f^{(r)} - q_n||_{W^{k-r,p}(I,\mu)} \\ &\leqslant c||f^{(r)} - q_n||_{W^{k-r,p}(I,\mu)}. \end{aligned}$$

We conclude that Q_n converges to f in $W^{k,p}(I, \mu)$. □

4. The case of non-compact support

Although the main interest in this section is the case of non-compact support, the following result can be applied to the case of compact support.

Theorem 4.1. *Let us consider* $1 \leqslant p < \infty$ and $\mu = (\mu_0, \ldots, \mu_k)$ *a p-admissible vectorial measure. Assume that there exist* $a \in \Delta$ *and a positive constant c such that*

$$c||g||_{W^{k,p}(\Delta,\mu)} \leqslant |g(a)| + |g'(a)| + \cdots + |g^{(k-1)}(a)| + ||g^{(k)}||_{L^p(\Delta,\mu_k)} \tag{4.1}$$

for every $g \in V^{k,p}(\Delta, \mu)$. *Then, P is dense in* $W^{k,p}(\Delta, \mu)$ *if and only if P is dense in* $L^p(\Delta, \mu_k)$.

Proof. We prove the non-trivial implication. Let us consider a fixed function $f \in V^{k,p}(\Delta, \mu)$. Assume that P is dense in $L^p(\Delta, \mu_k)$, and choose a sequence $\{q_n\}$ of

polynomials which converges to $f^{(k)}$ in $L^p(\Delta,\mu_k)$. Then the polynomials

$$Q_n(x) := f(a)+f'(a)(x-a)+\cdots+f^{(k-1)}(a)\frac{(x-a)^{k-1}}{(k-1)!} + \int_a^x q_n(t)\frac{(x-t)^{k-1}}{(k-1)!}\,dt$$

satisfy

$$c||f-Q_n||_{W^{k,p}(\Delta,\mu)} \leqslant ||f^{(k)}-Q_n^{(k)}||_{L^p(\Delta,\mu_k)} = ||f^{(k)}-q_n||_{L^p(\Delta,\mu_k)},$$

since $(f-Q_n)^{(j)}(a)=0$ for $0\leqslant j<k$, and we conclude that the sequence of polynomials $\{Q_n\}$ converges to f in $W^{k,p}(\Delta,\mu)$. □

We show now that Theorem 4.1 is very useful finding a wide class of measures satisfying (4.1). The following inequality can be found in [Mu2,M, p. 40].

Muckenhoupt inequality II. *Let us consider* $1\leqslant p<\infty$ *and* μ_0,μ_1 *measures in* $(0,\infty)$ *with* $w_1 := d\mu_1/dx$. *Then there exists a positive constant* c *such that*

$$\left|\left|\int_0^x g(t)\,dt\right|\right|_{L^p((0,\infty),\mu_0)} \leqslant c||g||_{L^p((0,\infty),\mu_1)}$$

for any measurable function g *in* $(0,\infty)$, *if and only if*

$$\sup_{r>0}\mu_0([r,\infty))||w_1^{-1}||_{L^{1/(p-1)}([0,r])}<\infty.$$

Remark. A similar result is true for the intervals (a,∞) and $(-\infty,a)$, with $a\in\mathbf{R}$.

Lemma 4.1. *Assume that* $w_0(x)\leqslant c_0x^{\alpha_0}e^{-\lambda x^\varepsilon}$ *and* $w_1(x)\geqslant c_1x^{\alpha_1}e^{-\lambda x^\varepsilon}$, *for* $x\geqslant A$, $w_0\in L^1([0,A])$, $w_1\in B_p([0,A])$, *with* $1\leqslant p<\infty$, $\lambda,\varepsilon,c_0,c_1,A>0$ *and* $\alpha_0,\alpha_1\in\mathbf{R}$. *If* $\alpha_0\leqslant\alpha_1+(\varepsilon-1)p$, *then* w_0,w_1 *satisfy Muckenhoupt inequality II.*

Proof. First of all observe that

$$(x^ae^{bx^\varepsilon})' = x^{a-1}e^{bx^\varepsilon}(a+b\varepsilon x^\varepsilon).$$

This implies $(x^ae^{bx^\varepsilon})'\asymp \operatorname{sign} b\,x^{a+\varepsilon-1}e^{bx^\varepsilon}$, as $x\to\infty$, if $b\neq0$. Therefore

$$\int_A^r x^ae^{bx^\varepsilon}dx\asymp r^{a+1-\varepsilon}e^{br^\varepsilon} \quad \text{if } b>0,$$
$$\int_r^\infty x^ae^{bx^\varepsilon}dx\asymp r^{a+1-\varepsilon}e^{br^\varepsilon} \quad \text{if } b<0$$

as $r\to\infty$. Hence, if $1<p<\infty$, we have as $r\to\infty$

$$\int_0^r w_1(x)^{-1/(p-1)}\,dx \asymp \int_A^r w_1(x)^{-1/(p-1)}\,dx \leqslant c\int_A^r x^{-\alpha_1/(p-1)}e^{\lambda x^\varepsilon/(p-1)}\,dx \asymp r^{1-\varepsilon-\alpha_1/(p-1)}e^{\lambda r^\varepsilon/(p-1)}.$$

If $p=1$, we have for big r

$$||w_1^{-1}||_{L^\infty([0,r])}\leqslant cr^{-\alpha_1}e^{\lambda r^\varepsilon}.$$

Consequently, we have for $1\leqslant p<\infty$

$$||w_1^{-1}||_{L^{1/(p-1)}([0,r])}\leqslant cr^{-\alpha_1+(1-\varepsilon)(p-1)}e^{\lambda r^\varepsilon}$$

as $r\to\infty$. Furthermore,

$$\int_r^\infty w_0(x)\,dx\leqslant c_0\int_r^\infty x^{\alpha_0}e^{-\lambda x^\varepsilon}dx\asymp r^{\alpha_0+1-\varepsilon}e^{-\lambda r^\varepsilon}.$$

The expression

$$\mu_0([r,\infty))||w_1^{-1}||_{L^{1/(p-1)}([0,r])}$$

is bounded for r in a compact set; it is bounded for big r, if

$$\lim_{r\to\infty} r^{\alpha_0+1-\varepsilon}e^{-\lambda r^\varepsilon}r^{-\alpha_1+(1-\varepsilon)(p-1)}e^{\lambda r^\varepsilon}<\infty.$$

This condition holds since $\alpha_0\leqslant\alpha_1+(\varepsilon-1)p$. □

Lemma 4.2. *Assume that* $w_0(x)\leqslant k_0x^{\beta_0}$ *and* $w_1(x)\geqslant k_1x^{\beta_1}$, *for* $0<x<b$, *with* $1\leqslant p<\infty$, $k_0,k_1>0$, $\beta_1\in\mathbf{R}$ *and* $\beta_0>-1$. *If* $\beta_0\geqslant\beta_1-p$, *then* w_0,w_1 *satisfy Muckenhoupt inequality I, with* $a=0$.

Proof. If $1<p<\infty$ and $\beta_1>p-1$, we have

$$\int_r^b w_1(x)^{-1/(p-1)}\,dx\leqslant c\int_r^b x^{-\beta_1/(p-1)}\,dx\asymp r^{1-\beta_1/(p-1)}.$$

If $p=1$ and $\beta_1>0$, we have

$$||w_1^{-1}||_{L^\infty([r,b])}\leqslant cr^{-\beta_1}.$$

Consequently, we have for $1\leqslant p<\infty$ and $\beta_1>p-1$

$$||w_1^{-1}||_{L^{1/(p-1)}([r,b])}\leqslant cr^{-\beta_1+p-1}.$$

Furthermore, since $\beta_0+1>0$, we obtain

$$\int_0^r w_0(x)\,dx\leqslant k_0\int_0^r x^{\beta_0}\,dx\asymp r^{\beta_0+1}.$$

If $\beta_1>p-1$, the expression

$$F(r):=\mu_0((0,r])||w_1^{-1}||_{L^{1/(p-1)}([r,b])},$$

is bounded for $r\in[\varepsilon,b]$ (with $\varepsilon>0$); it is bounded for $r\in(0,\varepsilon)$, if

$$\lim_{r\to 0^+} r^{\beta_0+1}r^{-\beta_1+p-1}<\infty.$$

This condition holds since $\beta_0\geqslant\beta_1-p$. If $\beta_1\leqslant p-1$, we obtain similarly that $F(r)$ is bounded since $\beta_0+1>0$ and

$$F(r)\leqslant cr^{\beta_0+1}\left(\log\frac{1}{r}\right)^{p-1}$$

for small r.

The following result is well known for $p=2$ (see e.g. [F, Chapter II]). It is also known for $p\neq 2$ but I have found no reference for it. I include a proof for the sake of completeness. □

Lemma 4.3. *Consider a scalar finite measure μ in* $\mathbf{R}$. *Assume that we have either*:

(1) *there exists $t>0$ such that $e^{t|x|}\in L^1(\mu)$,*
(2) *$\operatorname{supp}\mu\subseteq[0,\infty)$ and there exists $t>0$ such that $e^{t\sqrt{x}}\in L^1(\mu)$.*

Then, the polynomials are dense in $L^p(\mu)$ for $1\leqslant p<\infty$.

Proof. Denote by $P^p(\mu)$ the closure of P in $L^p(\mu)$. Consider hypothesis (1). We show first that if $g\in L^q(\mu)$ with $1<q\leqslant\infty$ and $\int x^n g(x)\,d\mu(x)=0$ for every $n\in\mathbf{N}$, then $g=0$ μ-almost everywhere. For such g, consider the Fourier transform F of $g\,d\mu$

$$F(z):=\int e^{ixz}g(x)\,d\mu(x).$$

Observe that we have

$$|e^{ixz}g(x)|\leqslant|g(x)|e^{|xIm\,z|}\leqslant|g(x)|e^{|x|t(q-1)/q},$$

if $|Im\ z|<t(q-1)/q$. Hölder inequality gives

$$\begin{aligned}\int|g(x)|e^{|x|t(q-1)/q}\,d\mu(x)&\leqslant||g||_{L^q(\mu)}||e^{|x|t(q-1)/q}||_{L^{q/(q-1)}(\mu)}\\&\leqslant||g||_{L^q(\mu)}||e^{t|x|}||_{L^1(\mu)}^{(q-1)/q},\end{aligned}$$

and this implies that F is a holomorphic function in $\{z\in\mathbf{C}:|Im\,z|<t(q-1)/q\}$. We also have for each $n\in\mathbf{N}$

$$F^{(n)}(z)=\int e^{ixz}(ix)^n g(x)\,d\mu(x),\quad F^{(n)}(0)=i^n\int x^n g(x)\,d\mu(x)=0.$$

Therefore F (the Fourier transform of $g\,d\mu$) is identically 0, and consequently $g=0$ μ-almost everywhere.

If $f\in L^p(\mu)\setminus P^p(\mu)$, then there exists $G\in(L^p(\mu))'$ such that $G(P)=0$ and $G(f)\neq 0$. Since μ is σ-finite, there exist $g\in L^q(\mu)$ with $q:=p/(p-1)>1$ and $G(h)=\int hg\,d\mu$ for every $h\in L^p(\mu)$. In particular, we have $G(x^n)=\int x^n g\,d\mu=0$ for every $n\in\mathbf{N}$. This fact implies $g=0$. Consequently $G(f)=0$ and we deduce $P^p(\mu)=L^p(\mu)$, for $1\leqslant p<\infty$. This finishes the proof if we assume hypothesis (1).

Consider now hypothesis (2). There exists a unique measure μ^* in $\mathbf{R}$ symmetric with respect to 0 and verifying $\int f(x)\,d\mu(x) = \int f(x^2)\,d\mu^*(x)$ for every $f \in L^1(\mu)$. Since μ^* satisfies (1), given any function $f \in L^p(\mu)$, there exists a sequence $\{h_n\} \subset P$ with $\lim_{n\to\infty} \int |f(x^2) - h_n(x)|^p d\mu^*(x) = 0$. Consequently, we have that there exists a sequence $\{H_n\} \subset P$ such that

$$\left(\int |f(x^2) - h_n(x)|^p d\mu^*(x)\right)^{1/p} \geqslant \left(\int \left| f(x^2) - \frac{h_n(x) + h_n(-x)}{2}\right|^p d\mu^*(x)\right)^{1/p}$$
$$= \left(\int |f(x^2) - H_n(x^2)|^p d\mu^*(x)\right)^{1/p} = \left(\int |f(x) - H_n(x)|^p d\mu(x)\right)^{1/p},$$

and this finishes the proof. □

These lemmas give the following results.

Proposition 4.1. *Consider $1 \leqslant p < \infty$ and a vectorial weight w in $(0, \infty)$, with*

(1) $w_j(x) \leqslant c_j x^{\beta_j}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k x^{\beta_k}$, *in* $(0, a)$,
(2) $w_j(x) \leqslant c_j x^{\alpha+(k-j)(\varepsilon-1)p} e^{-\lambda x^\varepsilon}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k x^\alpha e^{-\lambda x^\varepsilon}$, *in* (a, ∞),

where $\alpha \in \mathbf{R}$, $a, \varepsilon, \lambda, c_j > 0$ and $\beta_j > -1$, for $0 \leqslant j \leqslant k$. Then the polynomials are dense in $W^{k,p}([0, \infty), w)$ if they are dense in $L^p((0, \infty), w_k)$ and $\beta_j \geqslant \beta_k - (k-j)p$, for $0 \leqslant j < k$.

Proof. An induction argument with Lemma 4.1 in (a, ∞) instead of $(0, \infty)$, gives for $0 \leqslant j < k$ and $f \in V^{k,p}([a, \infty), w)$,

$$\int_a^\infty \left| f^{(j)}(x) - f^{(j)}(a) - \cdots - f^{(k-1)}(a)\frac{(x-a)^{k-j-1}}{(k-j-1)!}\right|^p w_j(x)\,dx \leqslant c\int_a^\infty |f^{(k)}(x)|^p w_k(x)\,dx,$$

and therefore

$$c||f^{(j)}||_{L^p((a,\infty),w_j)} \leqslant ||f^{(k)}||_{L^p((a,\infty),w_k)} + \sum_{i=j}^{k-1} |f^{(i)}(a)|$$

for $0 \leqslant j < k$ and $f \in V^{k,p}([a, \infty), w)$. Consequently, we have

$$c||f||_{W^{k,p}([a,\infty),w)} \leqslant ||f^{(k)}||_{L^p((a,\infty),w_k)} + \sum_{j=0}^{k-1} |f^{(j)}(a)| \tag{4.2}$$

for all $f \in V^{k,p}([a,\infty),w)$. If we use now Lemma 4.2 in $(0,a)$, a similar argument gives

$$c||f||_{W^{k,p}([0,a],w)} \leqslant ||f^{(k)}||_{L^p((0,a),w_k)} + \sum_{j=0}^{k-1} |f^{(j)}(a)| \tag{4.3}$$

for all $f \in V^{k,p}([0,a],w)$. Theorem 4.1, (4.2) and (4.3) give the proposition. □

We have the following immediate consequence of Proposition 4.1 and Lemma 4.3.

Corollary 4.1. *Consider $1 \leqslant p < \infty$ and a vectorial weight w, with $w_j(x) \asymp x^{\alpha_j} e^{-\lambda x^{\varepsilon}}$ in $(0,\infty)$, for $0 \leqslant j \leqslant k$, where $\varepsilon \geqslant 1/2$, $\lambda > 0$ and $\alpha_j > -1$, for $0 \leqslant j \leqslant k$. Then the polynomials are dense in $W^{k,p}([0,\infty),w)$ if $\alpha_k - (k-j)p \leqslant \alpha_j \leqslant \alpha_k + (k-j)(\varepsilon-1)p$, for $0 \leqslant j < k$.*

Proposition 4.2. *Consider $1 \leqslant p < \infty$ and a vectorial weight w in $\mathbf{R}$, with*

(1) $w_j(x) \leqslant c_j |x|^{\alpha+(k-j)(\varepsilon-1)p} e^{-\lambda|x|^{\varepsilon}}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k |x|^{\alpha} e^{-\lambda|x|^{\varepsilon}}$, *in* (B,∞),
(2) $w_j(x) \leqslant c_j |x|^{\alpha'+(k-j)(\varepsilon'-1)p} e^{-\lambda'|x|^{\varepsilon'}}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k |x|^{\alpha'} e^{-\lambda'|x|^{\varepsilon'}}$, *in* $(-\infty,-A)$,
(3) $w_j(x) \in L^1([-A,B])$, *for* $0 \leqslant j \leqslant k$, $w_k(x) \in B_p([-A,B])$,
where $\alpha, \alpha' \in \mathbf{R}$, $\varepsilon, \varepsilon' \geqslant 1$ and $A, B, \lambda, \lambda', c_j > 0$, for $0 \leqslant j \leqslant k$. Then the polynomials are dense in $W^{k,p}(\mathbf{R},w)$ if they are dense in $L^p(\mathbf{R},w_k)$.

Remark. The same result is true for $\varepsilon \geqslant 1/2$ if we change $\mathbf{R}$ by $(0,\infty)$.

Proof. The argument is similar to the one in Proposition 4.1, with 0 instead of a. In this case, we only use Lemma 4.1. □

We obtain the following consequence of Proposition 4.2 and Lemma 4.3.

Corollary 4.2. *Consider $1 \leqslant p < \infty$ and a vectorial weight w in $\mathbf{R}$, with $w_j(x) \asymp |x|^{\alpha_j} e^{-\lambda|x|^{\varepsilon}}$ in $\mathbf{R}$, for $0 \leqslant j \leqslant k$, where $\varepsilon \geqslant 1$, $\lambda > 0$ and $\alpha_j > -1$, for $0 \leqslant j \leqslant k$. Assume also that $\alpha_k < p-1$ if $p > 1$, and $\alpha_k \leqslant 0$ if $p = 1$. Then the polynomials are dense in $W^{k,p}(\mathbf{R},w)$ if $\alpha_j \leqslant \alpha_k + (k-j)(\varepsilon-1)p$, for $0 \leqslant j < k$.*

We can obtain similar results for weights of fast decreasing degree. The following results are not sharp since the sharp results are hard to write and do not involve any new idea.

Define inductively the functions $\exp_{\lambda_1,\ldots,\lambda_n}$ as follows:

$$\exp_{\lambda}(t) := \exp(\lambda t), \qquad \exp_{\lambda_1,\ldots,\lambda_n}(t) := \exp(\lambda_1 \exp_{\lambda_2,\ldots,\lambda_n}(t)).$$

Lemma 4.4. *Consider* $1 \leqslant p < \infty$ *and a scalar weight* $w(x) \asymp \exp_{-\lambda_1,\lambda_2,\ldots,\lambda_n}(x^\varepsilon)$ *in* $(0, \infty)$, *where* $n > 1$ *and* $\varepsilon, \lambda_1, \lambda_2, \ldots, \lambda_n > 0$. *Then* w, w *satisfy Muckenhoupt inequality II.*

Proof. A straightforward computation shows that the derivative of the function

$$x^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i,\lambda_{i+1},\ldots,\lambda_n}(x^\varepsilon)$$

converges to zero as $x \to \infty$. Now, if b is any non-zero real number, we have that

$$\frac{d}{dx}\left(\exp_{b,\lambda_2,\ldots,\lambda_n}(x^\varepsilon) x^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i,\lambda_{i+1},\ldots,\lambda_n}(x^\varepsilon)\right) \asymp \operatorname{sign} b \exp_{b,\lambda_2,\ldots,\lambda_n}(x^\varepsilon)$$

in $(1, \infty)$. Hence we have that

$$\int_r^\infty w \asymp \exp_{-\lambda_1,\lambda_2,\ldots,\lambda_n}(r^\varepsilon) r^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i,\lambda_{i+1},\ldots,\lambda_n}(r^\varepsilon),$$

$$||w^{-1}||^{1/(p-1)}_{L^{1/(p-1)}([0,r])} \asymp \exp_{\lambda_1/(p-1),\lambda_2,\ldots,\lambda_n}(r^\varepsilon) r^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i,\lambda_{i+1},\ldots,\lambda_n}(r^\varepsilon)$$

in $(1, \infty)$ if $1 < p < \infty$. Consequently, we have

$$||w^{-1}||_{L^{1/(p-1)}([0,r])} \asymp \exp_{\lambda_1,\lambda_2,\ldots,\lambda_n}(r^\varepsilon) \left(r^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i,\lambda_{i+1},\ldots,\lambda_n}(r^\varepsilon)\right)^{p-1}$$

in $(1, \infty)$ if $1 < p < \infty$. The result for $p = 1$ is trivial. Therefore

$$\left(\int_r^\infty w\right) ||w^{-1}||_{L^{1/(p-1)}([0,r])} \asymp \left(r^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i,\lambda_{i+1},\ldots,\lambda_n}(r^\varepsilon)\right)^p$$

in $(1, \infty)$ if $1 \leqslant p < \infty$. This finishes the proof, since $w \in L^1((0, \infty))$. □

Proposition 4.3. *Consider* $1 \leqslant p < \infty$ *and a vectorial weight* w, *with* $w_j(x) \leqslant c_j \exp_{-\lambda_1,\lambda_2,\ldots,\lambda_n}(|x|^\varepsilon)$ *in* $\mathbf{R}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k \exp_{-\lambda_1,\lambda_2,\ldots,\lambda_n}(|x|^\varepsilon)$ *in* $\mathbf{R}$, *where* $n > 1$ *and* $\varepsilon, \lambda_1, \lambda_2, \ldots, \lambda_n, c_0, c_1, \ldots, c_k > 0$. *Then the polynomials are dense in* $W^{k,p}(\mathbf{R}, w)$ *if they are dense in* $L^p(\mathbf{R}, w_k)$.

Remark. The same result is true if we change $\mathbf{R}$ by $(0, \infty)$.

Proof. It is enough to follow the argument in the proof of Proposition 4.1, using Lemma 4.4 instead of Lemmas 4.1 and 4.2. □

The following result is an immediate consequence of Proposition 4.3 and Lemma 4.3.

Corollary 4.3. *Consider* $1 \leqslant p < \infty$ *and a vectorial weight* w, *with* $w_j(x) \asymp \exp_{-\lambda_1,\lambda_2,\ldots,\lambda_n}(|x|^\varepsilon)$ *in* $\mathbf{R}$, *for* $0 \leqslant j \leqslant k$, *where* $n > 1$ *and* $\varepsilon, \lambda_1, \lambda_2, \ldots, \lambda_n > 0$. *Then the polynomials are dense in* $W^{k,p}(\mathbf{R}, w)$.

5. Comparable norms in generalized Sobolev spaces

It is important to know when two Sobolev norms are comparable. Here we prove a result on comparable Sobolev norms.

It is not difficult to see that if μ and v are σ-finite measures in a measurable space X such that $L^p(X,\mu) = L^p(X,v)$, then μ and v are comparable measures and the norms in $L^p(X,\mu)$ and $L^p(X,v)$ are also comparable (see Lemma 5.1).

This is not true for Sobolev spaces, as shows the following example: if μ is a finite vectorial measure in $[a,b]$ verifying $d\mu_k = dx$ and $\#\operatorname{supp}\mu_0 \geqslant k$, we have that $W^{k,p}([a,b],\mu) = W^{k,p}([a,b])$ (see [RARP1, Corollary 4.5]), and $\mu_0, \ldots, \mu_{k-1}$ may not be comparable to Lebesgue measure. However, we have also that the norms in these two spaces are comparable by Corollary 4.5 in [RARP1] or Theorem 5.1.

The following result generalizes this situation.

Theorem 5.1. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, \ldots, \mu_k)$, $v = (v_0, \ldots, v_k)$ σ-finite p-admissible vectorial measures, with absolutely continuous parts w, v, respectively. Assume that we have $W^{k,p}(\Delta,\mu) = W^{k,p}(\Delta,v)$ and the additional conditions*:

(1) $(\mu_j + v_j)(\Omega^{(j)}) < \infty$ *for* $0 \leqslant j < k$,
(2) $\Omega^{(j)} = \overline{\Omega_{j+1} \cup \cdots \cup \Omega_k}$ *and it is a finite union of bounded intervals, for* $0 \leqslant j < k$,
(3) $w_j^{-1}, v_j^{-1} \in L^{1/(p-1)}(A_j)$ *if* $A_j \neq \emptyset$, *where* A_j *is the open set* $A_j := \Omega_j \setminus \Omega^{(j)}$ *for* $0 < j < k$ *and* $A_k := \Omega_k$.

Then the norms in $W^{k,p}(\Delta,\mu)$ and $W^{k,p}(\Delta,v)$ are comparable.

Remark. The sets Ω_j can be distinct for w_j and v_j if $0 \leqslant j < k$, but the condition $W^{k,p}(\Delta,\mu) = W^{k,p}(\Delta,v)$ implies that $\Omega^{(j)}$ and A_j (if $0 \leqslant j \leqslant k$) are the same for w and v.

In order to prove Theorem 5.1 we need the following lemma.

Lemma 5.1. *Let us consider σ-finite measures μ, v in a measurable space X such that $L^p(X,\mu) = L^p(X,v)$ for some $1 \leqslant p < \infty$. Then μ and v are comparable measures, the norms in $L^p(X,\mu)$ and $L^p(X,v)$ are comparable and even $L^q(X,\mu) = L^q(X,v)$ for every $1 \leqslant q \leqslant \infty$.*

Proof. It is immediate that the sets with zero measure are the same for both measures. Therefore, μ and v are mutually absolutely continuous and we can write $d\mu = h\,dv$.

Assume that h is not comparable to 1 in X. Without loss of generality, we can assume that $h^{-1}((0,t])$ has positive v-measure for every $t > 0$. (The case $h^{-1}([t,\infty))$ has positive v-measure for every $t > 0$ is similar, changing the roles of μ and v.) Let us consider a decreasing sequence $\{t_n\}$ with limit 0 such that $0 < t_n < 1/n$ and $X_n := h^{-1}((t_{n+1}, t_n])$ has positive v-measure for every n, and an increasing sequence of

measurable sets A^n such that $X = \cup_n A^n$ and $0 < v(A^n) < \infty$ for every n. Let us define a sequence of measurable sets $Y_n := X_n \cap A^{m_n}$ where m_n are chosen in order to verify $0 < v(Y_n) < \infty$ for every n.

We define now the function

$$g := \sum_{n=1}^{\infty} (nv(Y_n))^{-1/p} \chi_{Y_n}.$$

Since $\{Y_n\}$ are pairwise disjoint, we have that

$$||g||^p_{L^p(v)} = \sum_{n=1}^{\infty} (nv(Y_n))^{-1} v(Y_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

and

$$\begin{aligned} ||g||^p_{L^p(\mu)} &= \sum_{n=1}^{\infty} (nv(Y_n))^{-1} \int_{Y_n} h \, dv \\ &\leqslant \sum_{n=1}^{\infty} (nv(Y_n))^{-1} t_n v(Y_n) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \end{aligned}$$

We have a contradiction since $g \in L^p(X, \mu) \setminus L^p(X, v)$. Consequently μ and v are comparable measures, the norms in $L^p(X, \mu)$ and $L^p(X, v)$ are comparable and $L^q(X, \mu) = L^q(X, v)$ for every $1 \leqslant q \leqslant \infty$. □

Proof of Theorem 5.1. Observe that $(\mu_j)_s(A_j) = (v_j)_s(A_j) = 0$ for $0 < j \leqslant k$, since μ and v are p-admissible measures. We show first that $\mu_j \asymp v_j$ in the set A_j for $0 < j \leqslant k$. This is immediate if $|A_j| = 0$; otherwise let us define $h_0 := (d\mu_j/dv_j)|_{A_j}$. Assume that h_0 is not comparable to 1 in A_j. Without loss of generality we can assume that $h_0^{-1}((0, t])$ has positive Lebesgue measure for every $t > 0$. (The case $h_0^{-1}([t, \infty))$ has positive Lebesgue measure for every $t > 0$ is similar, changing the roles of μ_j and v_j.)

Property (2) gives that A_j is a finite union of bounded intervals $I_j^1 \cup \cdots \cup I_j^N$. Therefore, there exists $1 \leqslant i \leqslant N$ such that $I_j^i \cap h_0^{-1}((0, t])$ has positive Lebesgue measure for every $t > 0$. Choose a proper subinterval $I_j \subset I_j^i$ such that $I_j \cap h_0^{-1}((0, t])$ has positive Lebesgue measure for every $t > 0$, and define $h := h_0|_{I_j}$.

Let us consider a decreasing sequence $\{t_n\}$ with limit 0 such that $0 < t_n < 1/n$ and $X_n := h^{-1}((t_{n+1}, t_n])$ has positive Lebesgue measure for every n, and an increasing sequence of measurable sets A_j^n such that $I_j = \cup_n A_j^n$ and $0 < v_j(A_j^n) < \infty$ for every n (recall that the remark to Definition 2 gives that the Lebesgue measure is absolutely continuous with respect to v_j in A_j). Let us define a sequence of measurable sets $Y_n := X_n \cap A_j^{m_n}$ where m_n are chosen in order to verify $0 < v_j(Y_n) < \infty$ for every n.

We define now the function

$$g := \sum_{n=1}^{\infty} (nv_j(Y_n))^{-1/p} \chi_{Y_n}.$$

Since $\{Y_n\}$ are pairwise disjoint, we have that

$$||g||^p_{L^p(v_j)} = \sum_{n=1}^{\infty} (nv_j(Y_n))^{-1} v_j(Y_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

and

$$\begin{aligned} ||g||^p_{L^p(\mu_j)} &= \sum_{n=1}^{\infty} (nv_j(Y_n))^{-1} \int_{Y_n} h\, dv_j \\ &\leqslant \sum_{n=1}^{\infty} (nv_j(Y_n))^{-1} t_n v_j(Y_n) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \end{aligned}$$

Observe that $g \in L^1(\mathbf{R})$:

$$||g||_{L^1(\mathbf{R})} = \int_{A_j} g w_j^{1/p} w_j^{-1/p} \leqslant ||g||_{L^p(\mu_j)} ||w_j^{-1}||^{1/p}_{L^{1/(p-1)}(A_j)} < \infty,$$

since we have (3) and $g \in L^p(\mu_j)$.

Recall that I_j is a proper subinterval of I_j^i. Then we have either $\sup I_j < \sup I_j^i$ or $\inf I_j^i < \inf I_j$. Assume that $\alpha := \sup I_j < \beta := \sup I_j^i$ (the other case is symmetric). Let us consider a function $\varphi \in C^\infty(\mathbf{R})$ with $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $(-\infty, \alpha]$ and $\varphi = 0$ in $[\beta, \infty)$. If we take $x_0 := \inf I_j^i$, then the function

$$G(x) := \varphi(x) \int_{x_0}^{x} g(t) \frac{(x-t)^{j-1}}{(j-1)!}\, dt$$

belongs to $C^{j-1}(\mathbf{R})$ and we have $\operatorname{supp} G \subseteq \bar{I}_j^i$ and $G = G' = \cdots = G^{(j)} = 0$ in $\mathbf{R} \setminus A_j$. Property (1) gives

$$\mu_i(A_j) \leqslant \mu_i(\Omega_j) \leqslant \mu_i(\Omega^{(i)}) < \infty \quad \text{for } 0 \leqslant i < j \leqslant k.$$

This fact and $g \in L^p(\mu_j)$ give $G \in W^{k,p}(\Delta, \mu)$. Since we have $g \notin L^p(v_j)$, we obtain that $G \notin W^{k,p}(\Delta, v)$, which is a contradiction with $W^{k,p}(\Delta, \mu) = W^{k,p}(\Delta, v)$. Therefore, we have proved that $\mu_j \asymp v_j$ in A_j.

Let us consider a function $f \in V^{k,p}(\Delta, \mu) = V^{k,p}(\Delta, v)$. We deduce by property (2) that $\Omega^{(0)} \setminus (\Omega_1 \cup \cdots \cup \Omega_k) = \partial(\Omega_1 \cup \cdots \cup \Omega_k)$ has only a finite number of points and then $(\Delta, \mu) \in \mathscr{C}$ (see Remark 1 to Definition 11). Therefore, we can apply part (b) of Theorem A and we have a representative f_0 in the same class than f in $W^{k,p}(\Delta, \mu) = W^{k,p}(\Delta, v)$ such that

$$\sum_{j=0}^{k-1} ||f_0^{(j)}||_{L^\infty(\Omega^{(j)})} \leqslant c ||f||_{W^{k,p}(\Delta,\mu)},$$

since we have property (2). Then we have by property (1)

$$\sum_{j=0}^{k-1} ||f^{(j)}||_{L^p(\Omega^{(j)}, v_j)} = \sum_{j=0}^{k-1} ||f_0^{(j)}||_{L^p(\Omega^{(j)}, v_j)} \leqslant c \sum_{j=0}^{k-1} ||f_0^{(j)}||_{L^\infty(\Omega^{(j)})} \leqslant c ||f||_{W^{k,p}(\Delta,\mu)}.$$

We have also

$$\sum_{j=1}^{k} ||f^{(j)}||_{L^p(A_j,v_j)} \leqslant c \sum_{j=1}^{k} ||f^{(j)}||_{L^p(A_j,\mu_j)} \leqslant c||f||_{W^{k,p}(\Delta,\mu)}.$$

Lemma 5.1 gives

$$||f||_{L^p(\Delta \setminus \Omega^{(0)},v_0)} \leqslant c||f||_{L^p(\Delta \setminus \Omega^{(0)},\mu_0)} \leqslant c||f||_{W^{k,p}(\Delta,\mu)}.$$

These inequalities give that there exists a positive constant, independent of f such that

$$||f||_{W^{k,p}(\Delta,v)} \leqslant c||f||_{W^{k,p}(\Delta,\mu)},$$

since we have

$$v_j(\mathbf{R} \setminus (\Omega_j \cup \Omega^{(j)})) = 0 \quad \text{for } 0<j<k,$$

$$v_0(\mathbf{R} \setminus \Delta) = 0, \quad v_k(\mathbf{R} \setminus A_k) = 0.$$

The reverse inequality is obtained by changing the roles of μ and v. □

References

[D] P.J. Davis, Interpolation and Approximation, Dover, New York, 1975.

[EL] W.N. Everitt, L.L. Littlejohn, The density of polynomials in a weighted Sobolev space, Rend. Mat. Ser. VII 10 (1990) 835–852.

[ELW1] W.N. Everitt, L.L. Littlejohn, S.C. Williams, Orthogonal Polynomials in Weighted Sobolev Spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 117, Marcel Dekker, New York, 1989, pp. 53–72.

[ELW2] W.N. Everitt, L.L. Littlejohn, S.C. Williams, Orthogonal polynomials and approximation in Sobolev spaces, J. Comput. Appl. Math. 48 (1993) 69–90.

[F] G. Freud, Orthogonal Polynomials, Pergamon Press, Oxford, 1971.

[GR] J. García-Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland, Amsterdam, 1985.

[HKM] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford Science Publ, Clarendon Press, Oxford, 1993.

[K] T. Kilpeläinen, Weighted Sobolev spaces and capacity, Ann. Acad. Sci. Fenn. Ser. A. I. Math. 19 (1994) 95–113.

[Ku] A. Kufner, Weighted Sobolev Spaces, Teubner Verlagsgesellschaft, Teubner-Texte zur Mathematik (Band 31), 1980; also published by Wiley, 1985.

[KO] A. Kufner, B. Opic, How to define reasonably weighted Sobolev spaces, Comment. Math. Univ. Caroline 25 (3) (1984) 537–554.

[KS] A. Kufner, A.M. Sändig, Some Applications of Weighted Sobolev Spaces, Teubner Verlagsgesellschaft, Teubner-Texte zur Mathematik (Band 100), 1987.

[LP] G. López Lagomasino, H. Pijeira, Zero location and n-th root asymptotics of Sobolev orthogonal polynomials, J. Approx. Theory 99 (1999) 30–43.

[M] V.G. Maz'ja, Sobolev Spaces, Springer, Berlin, 1985.

[Mu1] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165 (1972) 207–226.

[Mu2] B. Muckenhoupt, Hardy's inequality with weights, Studia Math. 44 (1972) 31–38.

[R1] J.M. Rodríguez, Weierstrass' Theorem in weighted Sobolev spaces, J. Approx. Theory 108 (2001) 119–160.

[R2] J.M. Rodríguez, The multiplication operator in Sobolev spaces with respect to measures, J. Approx. Theory 109 (2001) 157–197.

[RARP1] J.M. Rodríguez, V. Alvarez, E. Romera, D. Pestana, Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials I, preprint.

[RARP2] J.M. Rodríguez, V. Alvarez, E. Romera, D. Pestana, Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials II, Approx. Theory Appl. 18 (2002) 1–32.

[T] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, 1978.